Dynamics of Nonlinear Stochastic Systems
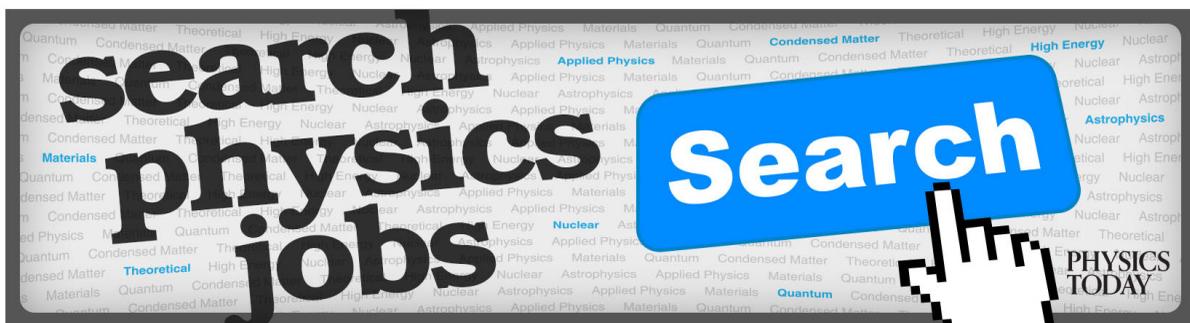
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Dynamics of Nonlinear Stochastic Systems*

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A method for treating nonlinear stochastic systems is described which it is hoped will be useful in both the quantum-mechanical many-body problem and the theory of turbulence. In this method the true problem is replaced by models that lead to closed equations for correlation functions and averaged Green's functions. The model solutions are exact descriptions of possible dynamical systems, and, as a result, they display certain consistency properties. For example, spectral components of Green's functions which must be positive-definite in the true problem automatically are so for the models. The models involve a new stochastic element: Random couplings are introduced among an infinite collection of similar systems, the true problem corresponding to the limit where these couplings vanish. The method is first applied to a linear oscillator with random frequency parameter. The mean impulse-response function of the oscillator is obtained explicitly for two successive models. The results suggest the existence of a sequence of model solutions which converges rapidly to the exact solution of the true problem. Applications then are made to the Schrödinger equation of a particle in a random potential and to Burgers' analog for turbulence dynamics. For both problems, closed model equations are obtained which determine the average Green's function, the amplitude of the mean field, and the covariance of the fluctuating field. The model solutions can be expressed as sums of infinite classes of terms from the formal perturbation expansions of the solutions to the true problems. It is suggested that correspondence to stochastic models may be a useful criterion to help judge the validity of partial summations of perturbation series.

1. INTRODUCTION

THIS paper is intended to introduce a method for treating certain problems where the dynamical equations are nonlinear in stochastic quantities. The quantum-mechanical many-body problem[1,2] and the theory of turbulence[3,4] are two fields of current interest where it is hoped that the method will prove useful. In such problems, there arise from the dynamical equations an infinite hierarchy of coupled equations which relate given ensemble averages to successively more complicated ones. An equivalent statement is that the prediction of a given average over a finite time requires the initial knowledge of an infinite number of averages. This situation, which commonly is called the closure problem, arises even when the nonlinear stochastic terms are linear in the dynamic variables. An example is linear wave propagation in a medium with random refractive index fluctuations.[5] Here the equation for the ensemble-averaged wave amplitude forms the base of an hierarchy involving successively higher cross-moments of the joint distribution of index and amplitude fluctuations.

A formal solution to the dynamical equations of any of the problems mentioned above may be obtained by treating the nonlinear terms as a perturbation and expanding by iteration.[3,6] One may then approximate statistical quantities by either truncating this expansion or summing tractable classes of terms to all orders. Another (and related) approach is to discard the cumulants of the statistical distribution above a certain order. Then all averages are expressible in terms of averages of this order and below, thereby providing a closure of the hierarchy of coupled statistical equations.[2,7]

In the method to be presented here, the true problem is replaced by models that lead, without approximation, to *closed* equations for correlation functions and averaged Green's functions. The model solutions are exact descriptions of possible dynamical systems, and, consequently, they have certain consistency properties which can be lacking in the approximation schemes mentioned. For example, spectral components of Green's functions which must be positive-definite in the true problem automatically are so in the models. A related property is that covariances satisfy certain realizability inequalities.

The models are constructed by introducing dynamical couplings among an infinite collection of similar systems, the true problem corresponding to the limit in which these couplings vanish. The coupling coefficients change randomly from one individual system in the collection to another. Thus they constitute a new stochastic element not present in the true problem. The models are most easily formulated in terms of a collective representation in which the variables are linear combinations of those of all the individual systems.

The closed statistical equations which characterize the models are obtained by averaging over an ensemble of realizations of the collection of coupled systems. When iteration expansions are generated for the averages of basic interest, it is found, using the collective representation, that the random couplings result in the cancellation of large classes of terms of all orders. The

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remaining terms are identical with corresponding ones in the expansion for the true problem (zero couplings). Although still of all orders, they have a sufficiently simple structure so that their sum represents the exact solution of closed integral equations.

The method of stochastic models is introduced in the present paper by application to a linear oscillator whose frequency parameter is Gaussianly distributed over an ensemble. This system has the virtue that it can be solved exactly. Furthermore, it displays great sensitivity to inadequacies in approximation schemes. Neither truncation of the perturbation series nor the cumulant-discard approach yields admissible approximations (Sec. 2). The collective representation and the general model are formulated in Secs. 3 and 4. Explicit solutions for the average impulse-response function of the oscillator then are obtained for two particular models (Secs. 5 and 7). They suggest the existence of a sequence of model solutions which converges rapidly to the exact solution for the true problem. In Sec. 8, model equations are obtained for the mean and covariance of the amplitude of the oscillator when driven by random forces. The generalization to non-Gaussian frequency distributions is described in Sec. 9.

In Sec. 6, approximations for the average response function are examined which represent infinite classes of terms in the perturbation expansion for the true problem, but which do *not* correspond to possible stochastic models. Although they are very plausible in terms of a diagrammatic representation of the perturbation series, these approximations have pathological characteristics. This suggests that correspondence to stochastic models may be a useful criterion to help judge the validity of partial summations of perturbation series in other analogous situations.

In Secs. 10 and 11, stochastic models are formulated for two problems of more physical interest: the Schrödinger equation of a particle in a random potential and Burgers' analog to turbulence dynamics. For both problems, closed integral equations are obtained which determine the average Green's function, the amplitude of the mean field, and the covariance of the fluctuating field. The models for these systems have an intimate formal relation to those for the random oscillator. In fact, the random potential problem is homologous to the oscillator problem, in the sense that the coupling coefficients characterizing corresponding models are identical in the two cases. Many results for the random potential problem can be obtained by inspection from the oscillator results. A comparison of the model equations for the random potential and turbulence problems illustrates the similarities and differences involved when the present method is applied to systems which are, respectively, linear and nonlinear in the dynamic variables.

In a paper to follow, stochastic models are formulated for classical and quantized nonlinear oscillators. Then the many-boson problem with interparticle forces is treated. This problem is homologous to the quantized nonlinear oscillator in the same way as the random potential problem is to the classical random linear oscillator. Particular attention is given to thermal equilibrium. The Einstein-Bose distribution law is derived by requiring equilibrium under arbitrary infinitesimal changes in the coupling among systems in a collection, without assuming a grand canonical or other particular distribution.

2. RANDOM OSCILLATOR

Let the amplitude $q(t)$ of a linear oscillator satisfy

$$dq(t)/dt + ibq(t) = 0, \qquad (2.1)$$

where b is a real time-independent parameter which is statistically distributed over an infinite ensemble of realizations of the oscillator. We shall be interested in determining the function $G(t) = \langle G_{[\ ]}(t) \rangle$, where $\langle \ \rangle$ denotes ensemble average and $G_{[\ ]}(t)$ is the response function[8] defined for $-\infty < t < \infty$ by

$$dG_{[\ ]}(t)/dt = -ibG_{[\ ]}(t), \quad G_{[\ ]}(0) = 1. \qquad (2.2)$$

We have, immediately,

$$G(t) = \langle \exp(-ibt) \rangle = \int_{-\infty}^{\infty} \exp(-ibt)P(b)db,$$

where $P(b)$ is the normalized probability density for b. Hence,

$$\tilde{G}(\omega) = P(\omega), \qquad (2.3)$$

where

$$\tilde{G}(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} G(t) \exp(i\omega t)dt.$$

Since $P(b) \geq 0$, $\tilde{G}(\omega)$ must satisfy the realizability condition

$$\tilde{G}(\omega) = |\tilde{G}(\omega)|. \qquad (2.4)$$

A particular consequence of Eq. (2.4) is

$$|G(t)| \leq G(0) = 1, \qquad (2.5)$$

which also follows from the fact that $|q(t)|$ is a constant of motion in each realization of the oscillator.

Now suppose that $P(b)$ is not known in closed form, but instead is specified by the infinite set of moments $\langle b \rangle$, $\langle b^2 \rangle$, $\langle b^3 \rangle$, $\cdots$. Then, by integrating Eq. (2.2) from 0 to t, iterating, and averaging, we may generate the formal solution.

$$G(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \langle b^n \rangle t^n/n!. \qquad (2.6)$$

Equation (2.6) corresponds precisely to the perturbation series for the averaged Green's function in certain statistical field physics problems. Let us explore its validity for the present problem by taking the

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[8] The reason for the peculiar bracket notation will become clear in Sec. 3.

example of Gaussian $P(b)$. Then we have

$$\tilde{G}(\omega) = (2\pi\langle b^2\rangle)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}\omega^2/\langle b^2\rangle), \qquad (2.7)$$

$$G(t) = \exp(-\tfrac{1}{2}\langle b^2\rangle t^2). \qquad (2.8)$$

But let us suppose that we do not know this closed form and instead are given the moment values

$$\langle b^n\rangle = 0 \quad (n \text{ odd}), \quad \langle b^{2n}\rangle = (\tfrac{1}{2}\langle b^2\rangle)^n 2n!/n!. \quad (2.9)$$

By Eq. (2.6) we have

$$G(t) = 1 + \sum_{n=1}^{\infty} (-\tfrac{1}{2}\langle b^2\rangle t^2)^n/n!, \qquad (2.10)$$

which, of course, is the power series expansion of Eq. (2.8).

The following observations may be made concerning Eq. (2.10). First, it is absolutely convergent for all t. Second, for $t > 2/\langle b^2\rangle^{\frac{1}{2}}$ the convergence rapidly becomes very poor so that very many terms must be taken to obtain a good approximation. Third, if the series is truncated after any finite number of terms, we have $G(t) \rightarrow \infty$, $t \rightarrow \infty$, in violation of the basic realizability condition (2.5). Thus, at no finite stage of the iteration treatment do we obtain an approximation with uniform validity for all t, and, in particular, at no stage does the spectral density $\tilde{G}(\omega)$ exist.

Let us next apply a second approximation scheme which has been widely used in statistical field physics. From Eq. (2.2) we may obtain the infinite set of coupled equations

$$dG(t)/dt = -i\langle bG_{[\,]}(t)\rangle \qquad G(0) = 1,$$

$$d\langle bG_{[\,]}(t)\rangle/dt = -i\langle b^2G_{[\,]}(t)\rangle \quad \langle bG_{[\,]}(0)\rangle = 0,$$

$$d\langle b^2G_{[\,]}(t)\rangle/dt = -i\langle b^3G_{[\,]}(t)\rangle \quad \langle b^2G_{[\,]}(0)\rangle = \langle b^2\rangle, \qquad (2.11)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

We may close off this hierarchy at successively higher stages by taking the zeroth approximation that b and $G_{[\,]}(t)$ are statistically independent, and then admitting successively higher-order cumulants of the joint distribution (higher "correlations" in the language of statistical field physics). Let us again assume Eq. (2.9). Then the appropriate successive closure approximations are

$$\langle bG_{[\,]}(t)\rangle = \langle b\rangle G(t) = 0,$$

$$\langle b^2G_{[\,]}(t)\rangle = \langle b^2\rangle G(t),$$

$$\langle b^3G_{[\,]}(t)\rangle = 3\langle b^2\rangle\langle bG_{[\,]}(t)\rangle, \qquad (2.12)$$

$$\langle b^4G_{[\,]}(t)\rangle = 6\langle b^2\rangle\langle b^2G_{[\,]}(t)\rangle - 3\langle b^2\rangle^2 G(t),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

[Note that if $G_{[\,]}(t)$ were statistically independent of b then all these relations would be exact.] On using these relations in turn to close off Eq. (2.11) at successively

higher stages, we obtain

$$G(t) = 1,$$

$$G(t) = \cos(b_*t),$$

$$G(t) = \tfrac{2}{3} + \tfrac{1}{3}\cos(\sqrt{3}b_*t),$$

$$G(t) = \tfrac{1}{6}(3+\sqrt{6})\cos[(3-\sqrt{6})^{\frac{1}{2}}b_*t] \qquad (2.13)$$

$$\qquad + \tfrac{1}{6}(3-\sqrt{6})\cos[(3+\sqrt{6})^{\frac{1}{2}}b_*t],$$

$$\cdot \quad \cdot \quad \cdot \quad ,$$

where $b_* = \langle b^2\rangle^{\frac{1}{2}}$.

Beyond the zeroth stage, which yields identical results in the two cases, the sequence (2.13) is distinctly superior to the approximations obtained by truncating Eq. (2.10). All members of the sequence satisfy Eqs. (2.4) and (2.5). In common with the iteration scheme, the first n even derivatives of $G(t)$ at $t=0$ are correct in the nth approximation. However, there still is no uniform validity in the sense $G(t) \rightarrow 0$, $t \rightarrow \infty$. None of the moments $\int_0^\infty t^nG(t)dt$ $(n=1, 2, \cdots)$ exist for any approximation in the sequence, whereas they all do for the exact solution. Alternatively, we may note that $\tilde{G}(\omega)$, which is smooth in the exact solution, is a sum of δ functions in any of the cumulant-discard approximations. The convergence to the exact $G(t)$ is still very poor for $t > 2/b_*$.

It is clear that the random oscillator exhibits in acute form certain shortcomings of the iteration (perturbation) and cumulant-discard approaches to dynamical equations which are nonlinear in stochastic quantities. Both for this reason and because of its simplicity, we shall use the random oscillator to illustrate the alternative approach which is the subject of this paper. The sensitivity to inadequacies in the method of approximation arises because the solution to the "unperturbed" equation, obtained by replacing the right-hand side of Eq. (2.2) with zero, has a monochromatic spectrum. In this respect, it resembles certain limiting cases of statistical field theory problems which are of current interest and to which our approach will be applicable. Examples are a quantum-mechanical particle in a random potential in the WKBJ limit, turbulence at infinite Reynolds number, and, in a less direct sense, a second-quantized many-boson system at very low temperature.

3. COLLECTIVE REPRESENTATION FOR A SET OF OSCILLATORS

We shall now describe a dynamical representation which is appropriate for formulating the stochastic models promised in Sec. 1. We shall introduce the representation formally and then give a physical interpretation and a comparison with more familiar concepts.

In Sec. 2 we treated an ensemble of realizations of a single oscillator. Now let us consider a collection of M oscillators $(M=2S+1$, $S=$positive integer) whose frequencies are identically and independently distributed over an ensemble of realizations of the collection. We

shall be interested in the limit $M \to \infty$, so that in reality we are introducing a kind of two-dimensional distribution. In place of the frequencies and amplitudes of the M individual oscillators, let us adopt the collective parameters and coordinates

$$b_\alpha = M^{-\frac{1}{2}} \sum_n \exp(i2\pi\alpha n/M)b_{[n]},$$
$$q_\alpha(t) = M^{-\frac{1}{2}} \sum_n \exp(i2\pi\alpha n/M)q_{[n]}(t), \quad (3.1)$$
$$(\alpha = 0, \pm 1, \cdots, \pm S) \quad (n = 1, 2, \cdots, M),$$

where $b_{[n]}$ and $q_{[n]}(t)$ are the frequency and amplitude of the nth oscillator.[9] The identities

$$M^{-1} \sum_\alpha \exp[i2\pi\alpha(n-m)/M] = \delta_{n,m},$$
$$M^{-1} \sum_n \exp[i2\pi n(\alpha-\beta)/M] = \delta_{\alpha,\beta} \quad (3.2)$$

yield

$$b_{[n]} = M^{-\frac{1}{2}} \sum_\alpha \exp(-i2\pi\alpha n/M)b_\alpha,$$
$$q_{[n]}(t) = M^{-\frac{1}{2}} \sum_\alpha \exp(-i2\pi\alpha n/M)q_\alpha(t). \quad (3.3)$$

Let us adopt hereafter the cyclic convention $\alpha \pm M \equiv \alpha$, which clearly is consistent with Eqs. (3.1) and (3.3).

From Eqs. (3.1) and (3.2) we easily find

$$b_{-\alpha} = b_\alpha^*, \quad \sum_\alpha b_\alpha b_{-\alpha} = \sum_n b_{[n]}^2,$$
$$\sum_\alpha q_\alpha(t)q_\alpha^*(t') = \sum_n q_{[n]}(t)q_{[n]}^*(t'), \quad (3.4)$$

where we have used $b_{[n]} = b_{[n]}^*$. From the equations of motion

$$dq_{[n]}(t)/dt + ib_{[n]}q_{[n]}(t) = 0, \quad (3.5)$$

we find

$$dq_\alpha(t)/dt = -iM^{-\frac{1}{2}} \sum_\beta b_\beta q_{\alpha-\beta}(t), \quad (3.6)$$

where $\alpha-\beta$ is to be interpreted according to the cyclic convention. This shows that the new coordinates, in contrast to the old, are dynamically coupled. By Eq. (3.4), $\sum_\alpha q_\alpha(t)q_\alpha^*(t)$ is a constant of motion. Let $G_{[n,m]}(t)$ denote the solution of Eq. (3.5) with $q_{[r]}(0) = \delta_{r,m}$ (all r) and let $G_{\alpha,\gamma}(t)$ denote the solution of Eq. (3.6) with $q_\mu(0) = \delta_{\mu,\gamma}$ (all μ). Since the individual oscillators are uncoupled, we have $G_{[n,m]}(t) = \delta_{n,m}G_{[n,n]}(t)$. Hence, by Eqs. (31.) and (3.3), and the linearity of the equations of motion,

$$G_{\alpha,\gamma}(t) = M^{-1} \sum_n \exp[i2\pi(\alpha-\gamma)n/M]G_{[n,n]}(t). \quad (3.7)$$

The functions $G_{\alpha,\gamma}(t)$ constitute the response matrix of the collection of oscillators in the new representation, and the $G_{[n,m]}(t)$ play this role in the old representation.

The statistical properties of the b_α are easily found from the assumption that the $b_{[n]}$ are identically and independently distributed. By Eqs. (3.1) and (3.2), we have immediately

$$\langle b_\alpha b_\beta \cdots b_\mu \rangle = 0 \quad (\alpha+\beta+\cdots+\mu \neq 0). \quad (3.8)$$

For Gaussian $b_{[n]}$ all odd-order moments vanish, and we

[9] Throughout this paper, quantities referring to individual systems will be labeled with square-bracketed Latin indices and collective quantities will be labeled with unbracketed Greek indices.

have

$$\langle b_{[n]}b_{[m]} \rangle = \delta_{n,m}\langle b^2 \rangle,$$
$$\langle b_{[n]}b_{[m]}b_{[r]}b_{[s]} \rangle$$
$$= (\delta_{n,m}\delta_{r,s} + \delta_{n,r}\delta_{m,s} + \delta_{n,s}\delta_{m,r})\langle b^2 \rangle^2, \quad (3.9)$$

$$\cdots,$$

where $\langle b^2 \rangle = \langle b_{[n]}^2 \rangle$, which is the same for all n. Then, by Eq. (3.1),

$$\langle b_\alpha \cdots b_\mu \rangle = 0 \quad \text{(odd number of factors)},$$
$$\langle b_\alpha b_\beta \rangle = \delta_{\alpha+\beta}\langle b^2 \rangle \quad (\delta_{\alpha+\beta} \equiv \delta_{0,\alpha+\beta}),$$
$$\langle b_\alpha b_\beta b_\gamma b_\epsilon \rangle = (\delta_{\alpha+\beta}\delta_{\gamma+\epsilon} + \delta_{\alpha+\gamma}\delta_{\beta+\epsilon} + \delta_{\alpha+\epsilon}\delta_{\beta+\gamma})\langle b^2 \rangle^2, \quad (3.10)$$

$$\cdots.$$

For any univariate $b_{[n]}$ distribution we find from Eq. (3.7),

$$\langle G_{\alpha,\beta}(t) \rangle = \delta_{\alpha,\beta}G(t),$$
$$\langle b_\beta G_{\gamma,\alpha}(t) \rangle = \delta_{\alpha,\beta+\gamma}M^{-\frac{1}{2}}\langle bG_{[\ ]}(t) \rangle,$$
$$\langle G_{\alpha,\gamma}(t)G_{\gamma,\alpha}^*(t) \rangle \quad (3.11)$$
$$= \delta_{\alpha,\gamma}|G(t)|^2 + M^{-1}[\langle |G_{[\ ]}(t)|^2 \rangle - |G(t)|^2].$$

Here $G_{[\ ]}(t)$ denotes $G_{[n,n]}(t)$, which is the same for all n, and $G(t) = \langle G_{[\ ]}(t) \rangle$, as in Sec. 2. In the limit $M \to \infty$, with which we are concerned, Eq. (3.11) gives

$$\langle G_{\alpha,\alpha}(t) \rangle = G(t), \quad \langle |G_{\alpha,\alpha}(t) - G(t)|^2 \rangle = O(M^{-1}),$$
$$\langle G_{\alpha,\gamma}(t)G_{\gamma,\alpha}^*(t) \rangle = O(M^{-1}) \quad (\alpha \neq \gamma), \quad (3.12)$$
$$\langle b_\gamma G_{\alpha-\gamma,\alpha}(t) \rangle = O(M^{-\frac{1}{2}}).$$

These relations show that the variance of

$$G_{\alpha,\alpha}(t) = M^{-1} \sum_n G_{[n,n]}(t)$$

vanishes in the limit. That is, $G_{\alpha,\alpha}(t)$ is statistically *sharp*. They further imply that the effective dynamical coupling between any given pair of degrees of freedom q_α and q_γ is infinitely weak in the limit. Equations (3.12) were obtained without explicit reference to Eq. (3.6), but their dynamical implications may also be inferred from the latter. The direct dynamical coupling of q_α to any q_γ arises from only one of the M terms, each $O(M^{-\frac{1}{2}})$, on the right-hand side of Eq. (3.6). Equations (3.12) show that the effective coupling is still $O(M^{-\frac{1}{2}})$ when the indirect interaction of q_α and q_γ through all the other degrees of freedom is included. The absence of fluctuations in $G_{\alpha,\alpha}(t)$ in the limit is consistent with the fact that this function is determined by the simultaneous interaction of q_α with an infinite number of other degrees of freedom; q_α exhibits negligible self-coupling, in contrast to $q_{[n]}$.

It is apparent, both from Eq. (3.1) and the convolution structure of Eq. (3.6), that the q_α have a close formal relation to Fourier coefficients. The physical significance of the new representation is best brought out, in fact, by a comparison with analysis into wave-number or frequency components. Let $\psi(x,t)$ be a scalar field, associated with an extended (one-dimensional) dynamical system, which is described by an

ensemble statistically invariant under translation (e.g., $\langle \psi(x,t)\psi(x',t)\rangle = \langle \psi(x+y,\ t)\psi(x'+y,\ t)\rangle$ for all y). The natural coordinates for describing the field then are wavenumber components, which change only by a phase factor under translation. Suppose, instead, the ensemble were invariant under time displacement. Then the natural coordinates would be frequency components.

Physical systems usually are neither statistically homogeneous nor stationary. However, if we form a collection of identically distributed individual systems, then obviously (and, it will appear at first sight, trivially) there is statistical invariance under permutation within the collection. The new representation is natural in the presence of this invariance in the same way that a wavenumber representation is natural when there is translational invariance. Actually, the permutation invariance is much broader than called for by strict analogy to translational invariance. Consequently, all the q_α ($\alpha \neq 0$) have identical statistical properties,[10] while, in general, the statistical properties of wavenumber components vary with wave number.

To examine the analogy further, let us take

$$\psi_k(t) = L^{-1}\int_{-L/2}^{L/2}\psi(x,t)\exp(-ikx)dx$$
$$(k = 2\pi\alpha/L,\ \alpha = 0,\ \pm 1,\ \cdots),$$

where we adopt the customary device of making the field cyclic with a period L which is as large as we wish compared to any relevant correlation length. Let us divide L into very many segments, each still very large compared to any correlation length. Then each segment contains a subsystem which has only a negligible statistical dependence on its neighbors. Furthermore, it is plausible to suppose that (over times which are not too large) each subsystem has only a negligible dynamical interaction with its neighbors. Then we validly may regard the set of subsystems as analogous to the collection of *perfectly* independent systems used above in defining the q_α. Considered in this way, the ψ_k and the q_α (for large M) play essentially similar roles. Both are linear combinations of the physical coordinates of a very large number of effectively independent systems.[11]

4. FORMULATION OF MODEL PROBLEMS

Consider, instead of Eq. (3.6), the more general equations

$$dq_\alpha(t)/dt = -iM^{-\frac{1}{2}}\sum_\beta \phi_{\alpha,\beta,\alpha-\beta}b_\beta q_{\alpha-\beta}(t),\quad (4.1)$$

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[10] The special role played by $\alpha = 0$ will become clear in Sec. 8.

[11] It is of interest to indicate how the q_α might be measured, in principle. Let a device sample each oscillator in the collection in turn, at time intervals τ, proceeding in order of increasing n and returning from the Mth oscillator to the first to repeat the cycle continuously. At each sampling instant let the device produce a sharp pulse, of strength proportional to $q_{[n]}(t)$. If τ is small enough that many cycles are completed before the $q_{[n]}$ change appreciably, the spectrum of the pulse train will approximate a line spectrum with frequencies $\alpha/M\tau$ and amplitudes proportional to the $q_\alpha(t)$.

where $\phi_{\alpha,\beta,\alpha-\beta}$ is independent of t and the same for every realization in the ensemble. We shall be interested in stochastic assignments of $\phi_{\alpha,\beta,\alpha-\beta}$ in the sense that this quantity will exhibit random changes in value as α and β are changed. By Eqs. (3.3) and (3.2), Eq. (4.1) implies

$$dq_{[n]}(t)/dt = -i\sum_{r,s}A_{[n,r,s]}b_{[r]}q_{[s]}(t),\quad (4.2)$$

where

$$A_{[n,r,s]} = M^{-2}\sum_{\beta,\gamma}\exp\{i2\pi[\beta(r-n) + \gamma(s-n)]/M\}\phi_{\beta+\gamma,\beta,\gamma}.\quad (4.3)$$

Thus, the individual oscillators in the collection now are dynamically coupled. When $\phi_{\alpha,\beta,\alpha-\beta} = 1$ for all α and β, then $A_{[n,r,s]} = \delta_{r,n}\delta_{s,n}$ so that we recover the original collection of uncoupled oscillators. The quantities $q_{[n]}(t)q_{[n]}{}^*(t)$ are no longer constants of motion in the general case. However, we shall require

$$\phi_{\alpha,\beta,\alpha-\beta} = \phi_{\alpha-\beta,-\beta,\alpha}{}^*.\quad (4.4)$$

Then, since $b_{-\beta} = b_\beta{}^*$, we find

$$d(\sum_n q_{[n]}q_{[n]}{}^*)/dt = d(\sum_\alpha q_\alpha q_\alpha{}^*)/dt = 0.\quad (4.5)$$

The response matrix corresponding to Eq. (4.1) satisfies

$$dG_{\alpha,\gamma}(t)/dt = -iM^{-\frac{1}{2}}\sum_\beta \phi_{\alpha,\beta,\alpha-\beta}b_\beta G_{\alpha-\beta,\gamma}(t),$$
$$G_{\alpha,\gamma}(0) = \delta_{\alpha,\gamma}.\quad (4.6)$$

Suppose that we carry out an iteration expansion of Eq. (4.6). The coefficient of t^n in the resulting power series for $G_{\alpha,\gamma}(t)$ is a sum over products of n factors ϕ and n factors b. It is clear, from the initial condition and the way the indices combine, that in each product the sum of the indices of the b factors must be $\alpha - \gamma$. Hence, by Eq. (3.8),

$$\langle G_{\alpha,\gamma}(t)\rangle = 0\quad (\alpha \neq \gamma).\quad (4.7)$$

By Eqs. (3.1), (3.3), and (4.7), we have

$$\langle G_{[n,m]}(t)\rangle$$
$$= M^{-1}\sum_\alpha \exp[-i2\pi(n-m)\alpha/M]\langle G_{\alpha,\alpha}(t)\rangle.\quad (4.8)$$

Therefore, if $\langle G_{\alpha,\alpha}(t)\rangle$ is independent of α, we have, by Eq. (3.2),

$$\langle G_{[n,m]}(t)\rangle = \delta_{n,m}G(t),\quad G(t) = \langle G_{\alpha,\alpha}(t)\rangle,\quad (4.9)$$

where $G(t)$ has now the same meaning as in Sec. 3. We shall be concerned only with ϕ assignments which yield Eq. (4.9), and, presently, we shall exhibit certain of their properties.

We wish now to develop an expression for $d\langle G_{\alpha,\alpha}(t)\rangle/dt$ by expanding the right-hand side of Eq. (4.6), with $\gamma = \alpha$. By writing

$$H_{\alpha-\beta,\beta,\alpha}(t) = -iM^{-\frac{1}{2}}\phi_{\alpha,\beta,\alpha-\beta}b_\beta G_{\alpha-\beta,\alpha}(t),\quad (4.10)$$

we may develop $\langle H_{\alpha-\beta,\beta,\alpha}(t)\rangle$ in powers of t by using the iteration expansion for $G_{\alpha-\beta,\alpha}(t)$. The coefficient of t^{r-1} in the resulting series is a sum, over the indices of all b factors except b_β, of products of r factors b and r

factors ϕ [including the factors b_β and $\phi_{\alpha,\beta,\alpha-\beta}$ which appear explicitly in Eq. (4.10)]. Let us take Gaussian $b_{[n]}$. (We shall return to the general case in Sec. 9.) Then, by Eqs. (3.8) and (3.10), only the odd powers of t survive and in the coefficients of these the indices of the b factors must be equal and opposite in pairs. In this way we find

$$\langle H_{\alpha-\beta,\beta,\alpha}(t)\rangle = M^{-1}\sum_{n=1}^{\infty}\sum_{p}(-1)^{n}C_{2n;p}(\alpha,\beta,\alpha-\beta)$$

$$\times\langle b^2\rangle^n t^{2n-1}/(2n-1)!, \quad (4.11)$$

where $M^{n-1}C_{2n;p}(\alpha,\beta,\alpha-\beta)$ is a sum of products of $2n$ factors ϕ and the index p ($p=1, 2, \cdots, 2n!/2^n n!$) labels (in an arbitrary order) the contributions which arise from all the possible pairings of the b factors.

The $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ through $n=2$ are

$$C_{2;1}(\ )=\phi_{\alpha,\beta,\alpha-\beta}\phi_{\alpha-\beta,-\beta,\alpha},$$

$$C_{4;1}(\ )=M^{-1}\sum_{\gamma}\phi_{\alpha,\beta,\alpha-\beta}\phi_{\alpha-\beta,-\beta,\alpha}\phi_{\alpha,\gamma,\alpha-\gamma}\phi_{\alpha-\gamma,-\gamma,\alpha},$$

$$C_{4;2}(\ )=M^{-1}\sum_{\gamma}\phi_{\alpha,\beta,\alpha-\beta}\phi_{\alpha-\beta,\gamma,\alpha-\beta-\gamma}$$

$$\times\phi_{\alpha-\beta-\gamma,-\gamma,\alpha-\beta}\phi_{\alpha-\beta,-\beta,\alpha}, \quad (4.12)$$

$$C_{4;3}(\ )=M^{-1}\sum_{\gamma}\phi_{\alpha,\beta,\alpha-\beta}\phi_{\alpha-\beta,\gamma,\alpha-\beta-\gamma}$$

$$\times\phi_{\alpha-\beta-\gamma,-\beta,\alpha-\gamma}\phi_{\alpha-\gamma,-\gamma,\alpha}.$$

The ϕ factors in Eqs. (4.12) other than $\phi_{\alpha,\beta,\alpha-\beta}$ are written from the right in the order in which they arise in the iteration process. (The values of p are assigned arbitrarily.)

Each $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ is closed in the sense that the initial index on any factor equals the final index on the factor to its immediate left [when ordered as in Eqs. (4.12)] while the middle indices are equal and opposite in pairs. This permits a systematic diagrammatic representation of the $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ and therefore of Eq. (4.11). With each $\phi_{\mu,\lambda,\sigma}$ or $\phi_{\mu,\lambda,\sigma}{}^*$ let us associate a vertex as shown in Fig. 1(a) or 1(b), respectively. The $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ then may be obtained by the following rules:

Connect $2n$ points by $2n$ solid line segments to form a single closed loop; then connect all the points in pairs by n dashed line segments to form a closed diagram of $2n$ vertices. Equip all the solid line segments with arrows pointing in the same sense. Choose one vertex, equip its dashed line segment with an ingoing arrow, and label its three line segments to correspond to $\phi_{\alpha,\beta,\alpha-\beta}$ in the sense of Fig. 1(a). Call this the fixed vertex and identify it by circling. Label the remaining dashed line segments γ, ϵ, $\cdots$ in any order and equip them with arrows (whose direction does not matter).[12] Complete the labeling of the solid line segments so that the sum of the indices labeling the ingoing lines equals the sum of those labeling the outgoing lines at every vertex. Now write the product of the ϕ factors associated with all the vertices according to Fig. 1(a), or according to Fig. 1(b) and Eq. (4.4). For each n there are $2n!/2^n n!$ distinct diagrams of this type, corresponding to the $2n!/2^n n!$ ways of connecting the vertices by dashed lines after one is chosen as the fixed vertex. Each diagram

Fig. 1. Vertices representing $\phi_{\mu,\lambda,\sigma}$ and $\phi_{\mu,\lambda,\sigma}{}^*$ for the random oscillator.

corresponds to one $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ (according to an arbitrary rule for assigning the values of p) and the latter is equal to M^{1-n} times the sum over γ, ϵ, $\cdots$ of the associated ϕ product.

The diagrams associated with Eqs. (4.12) are shown in Fig. 2.

It is clear from Eq. (4.11) that $\langle G_{\alpha,\alpha}(t)\rangle$ will be independent of α, and therefore Eq. (4.9) will hold, if

$$M^{-1}\sum_{\beta}C_{2n;p}(\alpha,\beta,\alpha-\beta)=C_{2n;p}, \quad (4.13)$$

where $C_{2n;p}$ is independent of α. In this case,

$$G(t)=1+\sum_{n=1}^{\infty}\sum_{p}(-1)^{n}C_{2n;p}\langle b^2\rangle^n t^{2n}/2n!. \quad (4.14)$$

We shall be concerned hereafter only with ϕ assignments such that Eq. (4.13) is satisfied when $M\to\infty$. The $C_{2n;p}$ may be interpreted as moments of the distribution of the quantity $\phi_{\mu,\lambda,\mu-\lambda}$ over the set of index values μ and λ.

Let us associate with $C_{2n;p}$ the diagram for $C_{2n;p}(\alpha,\beta,\alpha-\beta)$, but with index labels and dashed line arrows (which are now superfluous) omitted. By Eq. (4.13), we have $C_{2n;p}=M^{-2}\sum_{\alpha,\beta}C_{2n;p}(\alpha,\beta,\alpha-\beta)$. Recalling the cyclic convention $\mu\equiv\mu\pm M$ (any μ), we see that the summation in this expression is equivalent to one over all M values of all the indices labeling lines in the diagram, subject only to the sum condition at each vertex. Consequently, the expression is independent of which is the fixed vertex; its value depends only on the order and topology of the diagram.

It is important to point out that Eq. (3.12) is valid for general ϕ's satisfying Eq. (4.13). In particular, $G_{\alpha,\alpha}(t)$ is statistically sharp $(M\to\infty)$. In the original case (all ϕ's unity) this was so because q_α interacted simultaneously with all the other degrees of freedom, and negligibly with itself. These properties clearly characterize the general case also, provided the ϕ's are bounded as $M\to\infty$. The validity of Eq. (3.12) in the general case is easily demonstrated[13] for any power of t

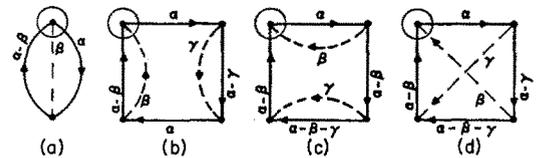
Fig. 2. Diagrams for $C_{2;1}(\ )$, $C_{4;1}(\ )$, $C_{4;2}(\ )$, and $C_{4;3}(\ )$.

[12] Reversal of the direction of the arrow on a dashed line labeled λ corresponds to the trivial notation change $\lambda\to-\lambda$ for the summed index λ in the associated $C_{2n;p}(\alpha,\beta,\alpha-\beta)$.

[13] See Appendix A. (a) Our "irreducible" diagrams are "proper" diagrams in the terminology of quantum field theory.

in the iteration expansion of the left sides of the equations, if one uses Eq. (3.10) and the fact that the ϕ's are the same for all realizations. We shall use the abbreviations C and $C(\ )$ to denote $C_{2n;p}$ and $C_{2n;p}(\alpha, \beta, \alpha-\beta)$, respectively, when it is not desired to specify particular subscripts and arguments.

Let us define a reducible C as one which may be factored into two or more C's of lower order, and an irreducible C as one which may not. Let us define a reducible $C(\ )$ as one which may be factored into the product of a lower-order $C(\ )$ with one or more C's, and an irreducible $C(\ )$ as one which may not. It follows that each reducible C is a product of irreducible C's and each reducible $C(\ )$ is a product of an irreducible $C(\ )$ with irreducible C's. It is easy to see from our rules that reducible $C(\ )$'s and C's (and only they) are associated with diagrams in which there is a part, or parts, connected to the rest of the diagram by only solid lines.[13a] Thus, $C_{2;1}$ and $C_{4;3}$ are irreducible, but $C_{4;1}$ and $C_{4;2}$ are reducible. By using Eq. (4.13) we find

$$C_{4;1} = C_{4;2} = (C_{2;1})^2.$$

Let us write each $C(\ )$ which appears in Eq. (4.11) as the product of an irreducible $C(\ )$ and irreducible C's, and then collect all the terms proportional to each irreducible $C(\ )$. We obtain a result of the form

$$\langle H_{\alpha-\beta,\beta,\alpha}(t)\rangle = M^{-1}\sum_n\sum_p{}^{\mathrm{irr}}(-1)^n$$
$$\times C_{2n;p}(\alpha, \beta, \alpha-\beta)\zeta_{2n;p}(t), \quad (4.15)$$

where $\sum^{\mathrm{irr}}$ denotes the sum over irreducible diagrams only. The $\zeta_{2n;p}(t)$ depend on the values of the irreducible C's but are independent of α and β. Each $\zeta_{2n;p}(t)$ contains all (odd) powers of t which are $\geq 2n-1$, since each $C_{2n;p}(\alpha, \beta, \alpha-\beta)$ appears in reducible $C(\ )$'s of all orders $\geq 2n$. The $\zeta_{2n;p}(t)$ turn out to have simple expressions in terms of $\langle b^2\rangle$ and $G(t)$ which may be found by comparing the explicit power series for $\zeta_{2n;p}(t)$ and $G(t)$. However, the same result may be obtained more transparently by a variational procedure which provides certain dynamical insights.

There are $M(M-1)$ ϕ's, and only M sums $\sum_\beta C_{2n;p}(\alpha, \beta, \alpha-\beta)$ for given n and p. In the limit $M\to\infty$ it will be possible, therefore, to make wide classes of variations $\Delta\phi$ such that Eqs. (4.4) and (4.13) continue to hold and such that $\Delta C_{2n;p}=0$ for all finite n. Under these constraints, $\Delta\zeta_{2n;p}(t)=0$. Consequently, we have

$$\Delta\langle H_{\alpha-\beta,\beta,\alpha}(t)\rangle = M^{-1}\sum_n\sum_p{}^{\mathrm{irr}}(-1)^n\zeta_{2n;p}(t)$$
$$\times\Delta C_{2n;p}(\alpha, \beta, \alpha-\beta). \quad (4.16)$$

Now consider a (finite) variation

$$\Delta\phi_{\alpha-\beta,-\beta,\alpha} = \Delta\phi_{\alpha,\beta,\alpha-\beta}{}^*$$

for a particular α and β, with all the other ϕ's fixed. We may vary the real and imaginary parts of $\phi_{\alpha-\beta,-\beta,\alpha}$ independently. Identical results are obtained by sup-

posing $\phi_{\alpha-\beta,-\beta,\alpha}$ to vary while $\phi_{\alpha-\beta,-\beta,\alpha}{}^* = \phi_{\alpha,\beta,\alpha-\beta}$ is held fixed, and we shall adopt the latter procedure. Then, by Eqs. (4.12),

$$\Delta C_{2;1}(\alpha, \beta, \alpha-\beta) = \phi_{\alpha,\beta,\alpha-\beta}\Delta\phi_{\alpha-\beta,-\beta,\alpha}, \quad (4.17)$$

while from Eq. (4.13) we see that $\Delta C_{2n;p} = O(M^{-1})$ for all n and p. Thus the constraints stated previously are satisfied for $M\to\infty$. By Eq. (4.6), $G_{\alpha-\beta,\alpha}(t)$ satisfies

$$dG_{\alpha-\beta,\alpha}(t)/dt$$
$$+iM^{-1}\sum_\sigma\phi_{\alpha-\beta,-\sigma,\alpha-\beta+\sigma}b_{-\sigma}G_{\alpha-\beta+\sigma,\alpha}(t) = 0. \quad (4.18)$$

The effect of the variation $\Delta\phi_{\alpha-\beta,-\beta,\alpha}$ is to produce on the right-hand side of Eq. (4.18) the additional term

$$-iM^{-\frac{1}{2}}\Delta\phi_{\alpha-\beta,-\beta,\alpha}b_{-\beta}G_{\alpha,\alpha}(t)$$

which, we note, is $O(M^{-\frac{1}{2}})$. Now we recall that $G_{\alpha-\beta,\alpha}(t)$ is simply the amplitude $q_{\alpha-\beta}(t)$ under a particular initial condition at $t=0$. Therefore, to order $M^{-\frac{1}{2}}$, we have

$$\Delta G_{\alpha-\beta,\alpha}(t) = \int_0^t G_{\alpha-\beta,\alpha-\beta}(t-s)$$
$$\times[-iM^{-\frac{1}{2}}\Delta\phi_{\alpha-\beta,-\beta,\alpha}b_{-\beta}G_{\alpha,\alpha}(s)]ds, \quad (4.19)$$

since $G_{\alpha-\beta,\alpha-\beta}$ is the diagonal response function for $q_{\alpha-\beta}$ and the perturbation does not affect the initial condition. [Note that $\Delta G_{\alpha,\alpha}(t)$ and $\Delta G_{\alpha-\beta,\alpha-\beta}(t)$ are $O(M^{-1})$ under our constraints.] On referring to Eq. (4.10), we obtain $\Delta\langle H_{\alpha-\beta,\beta,\alpha}(t)\rangle$ immediately. It is clear that our variation gives $\Delta C_{2n;p}(\alpha, \beta, \alpha-\beta) = O(M^{-1})$ for all irreducible diagrams with $n>1$. Then, by Eqs. (4.16) and (4.17), we find

$$\zeta_{2;1}(t) = \int_0^t\langle b_\beta b_{-\beta}G_{\alpha-\beta,\alpha-\beta}(t-s)G_{\alpha,\alpha}(s)\rangle ds \quad (4.20)$$

in the limit $M\to\infty$. As we have noted previously, $G_{\alpha,\alpha}(t)$ and $G_{\alpha-\beta,\alpha-\beta}(t)$ are statistically sharp in the limit. Therefore, by Eqs. (4.9) and (3.10),

$$\zeta_{2;1}(t) = \langle b^2\rangle\int_0^t G(t-s)G(s)ds. \quad (4.21)$$

The higher $\zeta_{2n;p}$ may be found by similar analysis based on more general variations. The result is

$$\zeta_{2n;p} = \langle b^2\rangle^n G(*G)^{2n-1}, \quad (4.22)$$

where $G(*G)^{2n-1}$ is a repeated convolution; e.g. (for argument t),

$$G(*G)^3 = \int_0^t ds\int_0^s ds'\int_0^{s'} ds''G(t-s)G(s-s')$$
$$\times G(s'-s'')G(s'').$$

On collecting the appropriate relations, we have the

final result

$$dG/dt = \sum_{n=1}^{\infty} \sum_{p}^{\mathrm{irr}} (-1)^n C_{2n;p} \langle b^2 \rangle^n G(*G)^{2n-1},$$

$$G(0) = 1. \quad (4.23)$$

The value of this infinite-series integro-differential equation for $G(t)$ is that only the irreducible $C_{2n;p}$ appear explicitly.

5. RANDOM COUPLING MODEL

We shall now consider a particular stochastic assignment of the ϕ's. Let

$$\phi_{\alpha,\beta,\alpha-\beta} = \exp(i\theta_{\alpha,\beta,\alpha-\beta}), \quad (5.1)$$

where $\theta_{\alpha,\beta,\alpha-\beta}$ is real and satisfies

$$\theta_{\alpha-\beta,-\beta,\alpha} = -\theta_{\alpha,\beta,\alpha-\beta}. \quad (5.2)$$

For each choice of α and β, let $\theta_{\alpha,\beta,\alpha-\beta}$ take a value at random in the interval 0 to 2π, subject only to Eq. (5.2). The value must be the same, of course, for every realization in the ensemble. Now let $M \to \infty$. Clearly this assignment satisfies Eq. (4.4). In addition, it yields $|\phi_{\alpha,\beta,\alpha-\beta}| = 1$, and, therefore, retains unaltered the *strengths* of the individual dynamical couplings of pairs q_α, $q_{\alpha-\beta}$ which characterize Eq. (3.6). Now, however, the *phases* of the couplings are completely unrelated for different pairs. We shall call the present choice the random coupling model.[14]

By referring to Eqs. (4.12) and (4.13), we find $C_{2;1} = 1$, as in the true problem (all ϕ's unity). Consider $C_{4;3}$, however. Each product in the sum has modulus 1, but the phase of the product changes at random with β and γ. Consequently, $C_{4;3} = O(M^{-1})$. In a similar fashion, we see that the only $C_{2n;p}$ which survive in the limit are those in which the product of ϕ's consists entirely of conjugate pairs and which, therefore, are reducible to powers of $C_{2;1}$. Consequently, for $M \to \infty$,

$$C_{2;1} = 1,$$
$$C_{2n;p} = 0 \quad (n > 1) \quad \text{(all irreducible diagrams).} \quad (5.3)$$

It follows that Eq. (4.23) reduces to the closed form

$$dG/dt + \langle b^2 \rangle G*G = 0, \quad G(0) = 1. \quad (5.4)$$

Equation (5.4) is readily solved by Laplace transformation. We find

$$G(t) = J_1(2b_*t)/b_*t, \quad (5.5)$$

$$\tilde{G}(\omega) = (\pi b_*)^{-1}[1 - (\omega/2b_*)^2]^{\frac{1}{2}} \quad (|\omega| \leq 2b_*),$$
$$= 0 \quad (|\omega| > 2b_*), \quad (5.6)$$

where $b_* = \langle b^2 \rangle^{\frac{1}{2}}$ as before. Considered as an approximation to Eqs. (2.8) and (2.7), the present results display a type of uniform validity which is absent in any finite

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[14] This model is unrelated to the "random phase approximation" [D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952)]. We make no assumption about the phases of the q_α themselves.
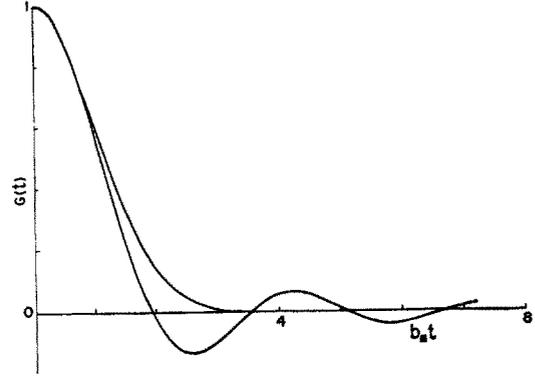
FIG. 3. Comparison of $G(t)$ for the true problem and the random coupling model.

stage of the iteration or cumulant-discard schemes discussed in Sec. 2. The spectral density of Eq. (5.6) is continuous, and Eqs. (2.4) and (2.5) are satisfied. All the derivatives of $\tilde{G}(\omega)$ exists at $\omega = 0$ so that all the moments $\int_0^\infty t^n G(t)dt$ exist. Equations (2.8) and (5.5) are compared in Fig. 3.

It is important that certain of the properties just listed could have been predicted from the sole fact that Eq. (5.4) is an exact equation ($M \to \infty$) for a realizable ϕ assignment satisfying Eq. (4.4), and, hence, for a conservative dynamical problem. We recall that $G_{\alpha,\alpha}(t) = q_\alpha(t)$ when $q_\gamma(0) = \delta_{\alpha,\gamma}$ (all γ). But then

$$\sum_\gamma q_\gamma(0)q_\gamma^*(0) = 1,$$

and, since $\sum_\gamma q_\gamma(t)q_\gamma^*(t)$ is a constant of motion, we have $|G_{\alpha,\alpha}(t)| \leq 1$, whence Eq. (2.5) readily follows. To establish Eq. (2.4), let us make, for each realization, a similarity transformation

$$q_\alpha(t) = \sum_\gamma B_{\alpha\gamma}q_\gamma'(t), \quad q_\gamma'(t) = \sum_\alpha B_{\gamma\alpha}^{-1}q_\alpha(t), \quad B_{\gamma\alpha}^{-1} = B_{\alpha\gamma}^*$$

($B_{\alpha\gamma}$ independent of t) such that Eq. (4.1) is brought to the diagonal form

$$dq_\gamma'(t)/dt + i\omega_\gamma q_\gamma'(t) = 0. \quad (5.7)$$

Since $\sum_\gamma q_\gamma' q_\gamma'^* = \sum_\alpha q_\alpha q_\alpha^*$ is a constant of motion, the ω_γ are real. Now if $G_{\gamma,\epsilon}'(t)$ is the response matrix of the new variables, $G_{\gamma,\epsilon}'(t) = \delta_{\gamma,\epsilon} \exp(-i\omega_\gamma t)$, and, therefore,

$$G_{\alpha,\alpha}(t) = \sum_{\gamma,\epsilon} B_{\alpha\gamma}G_{\gamma,\epsilon}'(t)B_{\epsilon\alpha}^{-1} = \sum_\gamma B_{\alpha\gamma}B_{\alpha\gamma}^* \exp(-i\omega_\gamma t).$$

Hence $\tilde{G}_{\alpha,\alpha}(\omega)$ is real and nonnegative in each realization, which implies Eq. (2.4). Finally, we note that the model problem resulting from a general realizable ϕ assignment involves the interaction of an infinite number of degrees of freedom q_α when $M \to \infty$. From this, and the fact that the b_α are continuously distributed over the ensemble, we may anticipate that $\tilde{G}(\omega)$ exhibits a continuous or band structure and that $G(t) \to 0$, $t \to \infty$.

It is possible to understand qualitatively why the complex detailed dynamics of the random coupling

model lead to a simplification of the statistical dynamics and to closure of Eq. (4.23). The function $\langle G_{\alpha,\alpha}(t) \rangle$ describes the decay of q_α due to transfer of an initial excitation, amplitude $q_\alpha(0) = 1$, to the rest of the degrees of freedom. In general, the decay requires that certain phase relations be set up between q_α and the other amplitudes. A phase relation between q_α and $q_{\alpha-\beta}$ can arise either from direct dynamical coupling (involving the coupling coefficients $-iM^{-\frac{1}{2}}\phi_{\alpha,\beta,\alpha-\beta}b_\beta$ and $-iM^{-\frac{1}{2}}\phi_{\alpha-\beta,-\beta,\alpha}b_{-\beta}$) or from indirect coupling through chains of other modes q_γ. In fact, each term in the irreducible diagram expansion Eq. (4.15) may be regarded as describing the transfer of excitation from q_α to $q_{\alpha-\beta}$ along the chain of intermediate modes represented by the directed solid line segments in the associated diagram. The closing of the solid line on itself then represents the reaction on mode α and the consequent diminution of q_α. The factors G, whose repeated convolution yields the $\zeta_{2n;p}$, incorporate the effect of the dynamical interaction as a whole on the transfer process. This effect is to relax the phase relations set up along the chain.

In the random coupling model, only the direct interaction, associated with $C_{2;1}$, is effective in the transfer of excitation. The contributions associated with the indirect paths of interaction cancel, when summed over all possible intermediate modes, because of the random phases of the ϕ's. The coupling of q_α and $q_{\alpha-\beta}$ to all the rest of the modes, therefore, affects $\langle H_{\alpha-\beta,\beta,\alpha}(t) \rangle$ only by relaxing the phase relations induced by the direct interaction of these two modes.

All $C_{2n;p}$ which are expressible as powers of $C_{2;1}$ have the value unity in the random coupling model, and all other $C_{2n;p}$ vanish. Thus we see from Eq. (4.14) that the power series for the model $G(t)$ consists of a particular subset of terms of all orders from the corresponding series for the true problem (all $C_{2n;p}$ unity). The terms retained are all those whose associated diagrams can be reduced to that for $C_{2;1}$ (Fig. 2) by by iterating any number of times, on any solid lines, the contraction operation shown in Fig. 4(a). Examples of included diagrams are shown in Fig. 4(b). It follows readily from Eq. (5.5) that the number of diagrams of this type with $2n$ vertices is

(a)

(b)
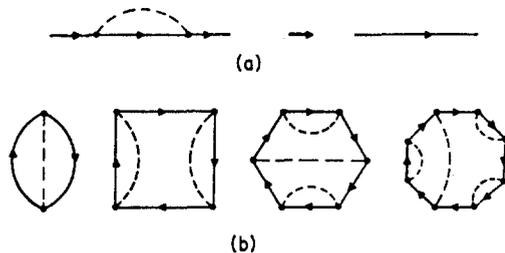
Fig. 4. (a) Contraction operation for the random coupling model; (b) typical diagrams included in the random coupling model. (Any vertex may be taken as the fixed vertex.)

$$(-1)^n \left\{ \frac{d^{2n}}{(dt)^{2n}} \left[\frac{J_1(2t)}{t} \right] \right\}_{t=0}.$$

6. INADMISSIBLE HIGHER APPROXIMATIONS

For the true problem (all $C_{2n;p} = 1$), Eq. (4.23) reads

$$dG/dt = \sum_{n=1}^{\infty} (-1)^n S_{2n} \langle b^2 \rangle^n G(*G)^{2n-1}, \quad G(0) = 1, \quad (6.1)$$

where S_{2n} is the number of irreducible diagrams of order $2n$. The first few S_{2n} are $S_2 = 1$, $S_4 = 1$, $S_6 = 4$, $S_8 = 27$.[15] The relative success of Eq. (5.4) as an approximation to Eq. (6.1) suggests that we seek higher approximations satisfying equations of the form

$$dG/dt = \sum_{n=1}^{R} (-1)^n S_{2n} \langle b^2 \rangle^n G(*G)^{2n-1},$$
$$G(0) = 1 \quad (R > 1), \quad (6.2)$$

which we obtain by giving all irreducible $C_{2n;p}$ the value one, $n \leq R$, and the value zero, $n > R$. One property of Eq. (6.2) can be predicted immediately. We recall that Eq. (4.23) represents simply a consolidation of Eq. (4.14). The present sums $\sum_p C_{2n;p}$ (reducible and irreducible diagrams included) clearly do not exceed the corresponding sums in the true problem. Since Eq. (2.10), which constitutes Eq. (4.14) for that problem, is absolutely convergent for all t, it follows that the power series expansions of the solutions of Eq. (6.2) are absolutely convergent for all t.

Nevertheless, these solutions are not valid higher approximations to Eq. (2.8). The reason is that the functions to which their expansions converge become infinite as $t \rightarrow \infty$. None of them, therefore, constitutes a uniform approximation, and for none of them does $\tilde{G}(\omega)$ exist. We shall illustrate this for $R = 2$. If $\mathbf{G}(p)$ denotes the Laplace transform of $G(t)$, then Eq. (6.2) for this case is equivalent to

$$p\mathbf{G}(p) = 1 - \langle b^2 \rangle [\mathbf{G}(p)]^2 + \langle b^2 \rangle^2 [\mathbf{G}(p)]^4. \quad (6.3)$$

Let us assume tentatively that $\mathbf{G}(0) = \int_0^\infty G(t) dt$ is finite. Then

$$[\mathbf{G}(0)]^2 = [1 \pm (-3)^{\frac{1}{2}}]/2\langle b^2 \rangle, \quad (6.4)$$

which is inconsistent with the reality of $G(t)$. Therefore, $\int_0^\infty G(t) dt$ cannot be finite. Further analysis readily shows that, for real ω, Eq. (6.3) is inconsistent with Re $\{\mathbf{G}(-i\omega)\} = 0(\omega^{-r})$, $\omega \rightarrow 0$, if r is any finite power. It follows that $G(t)$ grows faster than any power of t as $t \rightarrow \infty$.

The numerical solutions of Eq. (6.2) for several values of R are compared with Eq. (2.8) in Fig. 5. As R increases, it will be noted that the approximations increase in accuracy for small t but diverge faster at large t. In this respect, our present results resemble very closely those of truncating the original power series
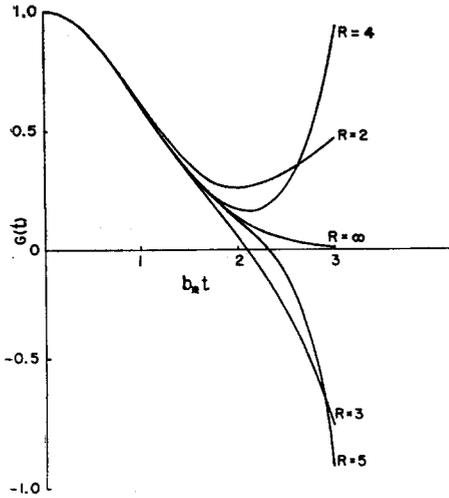
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[15] In general, $S_{2n} \neq (n-1)^{n-1}$.

FIG. 5. Solutions of Eq. (6.2) compared with $G(t)$ for the true problem $(R=\infty)$.

Eq. (2.10) after a finite number of terms. They do not appear to represent a significant improvement over the latter.

The failure of the present approximations has an immediate interpretation. Divergence of $G(t)$ as $t \to \infty$ is inconsistent with Eq. (2.5). It follows that the values of the irreducible C's implied by Eq. (6.2) are not realizable by any assignment of the ϕ's consistent with Eq. (4.4). Thus these approximations do not correspond to any dynamical model in our sense.[16]

If we regard a stochastic assignment of the $\phi_{\alpha,\beta,\alpha-\beta}$ for $M \to \infty$ as a distribution over the set of index values α and β, then the $C_{2n;p}$ are moments of this distribution and there are an infinite set of realizability inequalities which they must satisfy. The values $C_{2;1}=1$, $C_{2n;p}=0$ (all higher irreducible diagrams) for the random coupling model correspond to complete statistical independence of the $\phi_{\alpha,\beta,\alpha-\beta}=\phi_{\alpha-\beta,-\beta,\alpha}{}^*$ in this sense. Nonvanishing values for the higher irreducible C's imply statistical correlation among the ϕ's.

The nature of realizability inequalities for simpler statistical problems suggests that, when $C_{2;1}=1$, a wide choice of realizable nonzero values can be given to the higher irreducible $C_{2n;p}$, provided these values are small enough. Let us consider the assignment

$$C_{2;1}=1, \quad C_{4;3}=a, \quad C_{2n;p}=0$$

(all higher irreducible diagrams), (6.5)

where a is a real constant. For $a=1$ this gives Eq. (6.2), $R=2$. Instead of Eq. (6.3), we now find, in the general case,

$$p\mathbf{G}(p)=1-\langle b^2\rangle[\mathbf{G}(p)]^2+a\langle b^2\rangle^2[\mathbf{G}(p)]^4, \quad (6.6)$$

whence

$$[\mathbf{G}(0)]^2=[1-(1-4a)^{\frac{1}{2}}]/2a\langle b^2\rangle. \quad (6.7)$$

Equation (6.6) yields real, nonnegative $\tilde{G}(\omega)$ for all ω if

$$-\tfrac{1}{12}\leq a\leq\tfrac{1}{4}, \quad (6.8)$$

which suggests that Eq. (6.8) may represent the range of realizability of Eq. (6.5). In Fig. 6, the solution $\tilde{G}(\omega)=\pi^{-1}\operatorname{Re}\{\mathbf{G}(-i\omega)\}$ on the relevant branch of Eq. (6.6) is compared, for several values of a, with Eqs. (2.7) and (5.6). It will be noted that the form of $\tilde{G}(\omega)$ changes continuously with a up to the limit $a=\tfrac{1}{4}$, where the slope at $\omega=0$ changes abruptly from 0 to ∞. It is apparent that none of the present approximations represents a substantial improvement over the random coupling solution $(a=0)$. For $a<0$, the form of $\tilde{G}(\omega)$ changes continuously with a down to the limit $a=-\tfrac{1}{12}$; there, a singularity appears at the cutoff point of the spectrum. We conclude tentatively, lacking contrary evidence, that Eq. (6.8) does represent the range of realizability of Eq. (6.5).

The results of the present section suggest that great caution be exercised in carrying out partial summations of diagrams in the power series expansion for $G(t)$. It is by no means true that the more terms summed, the better the approximation. Our inadmissible approximation Eq. (6.2) $(R=2)$ is equivalent to the retention, in Eq. (4.14) for the true problem, of all terms whose diagrams can be reduced to the diagram for $C_{2;1}$ by iterated application, on any solid lines, of the contraction operations shown in Fig. 7(a). Examples are shown in Fig. 7(b). Thus, the terms retained are selected according to well-defined and plausible topological properties of the diagrams. Moreover, as we have noted, in the t domain they constitute an absolutely convergent subseries of an absolutely convergent series.

It will have been recognized by this point that the diagram summations we have employed are intimately

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[16] The relations between irreducible and reducible C's, to which we have appealed in discussing Eq. (6.2), are not affected by the unrealizability of the C's. These relations may be regarded here as formal implications of the requirement that Eq. (4.14) agree with the power series for $G(t)$ obtained by the iteration solution of Eq. (4.23).
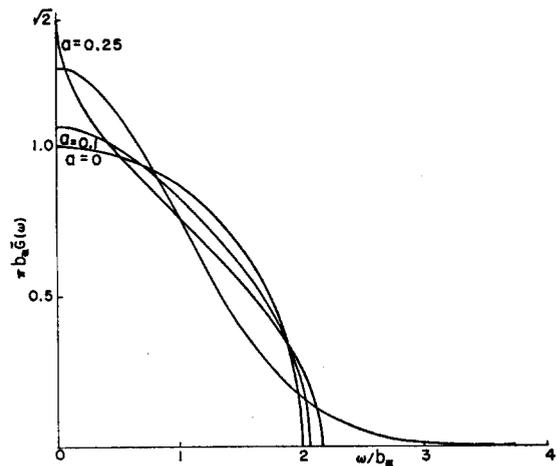
FIG. 6. Solutions $\tilde{G}(\omega)$ of Eq. (6.6) for several values of a compared with $\tilde{G}(\omega)$ for the true problem.
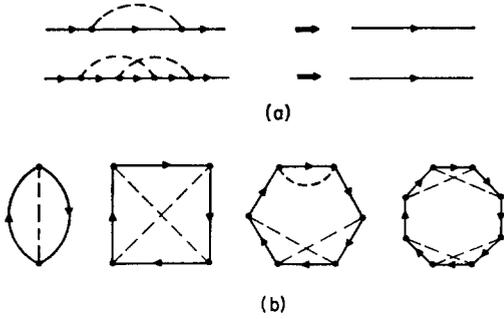
FIG. 7. (a) Contraction operations for Eq. (6.2) with $R=2$; (b) typical diagrams included by Eq. (6.2) with $R=2$.

related to summations of perturbation series terms in quantum field theory and quantum statistical mechanics. Our present results suggest that caution be exercised in these problems also. There too, it is possible that plausible-appearing and summable classes of diagrams are better omitted than included. We hope to return to these questions in a later paper.

7. SECOND STOCHASTIC MODEL

The results of the last section emphasize the desirability of seeking higher approximations to Eq. (6.1) which correspond to realizable values of the C's. We shall now describe a second stochastic model for which $\tilde{G}(\omega)$ satisfies Eq. (2.4) and is substantially closer to Eq. (2.7) than is the random coupling result. Consider the contraction operation shown in Fig. 8(a). Each application of it to a diagram reduces the number of vertices by two. Let us take $C_{2;1}=1$ and assign the value a^{n-1} to all irreducible $C_{2n;p}$ whose diagrams can be transformed into that for $C_{2;1}$ by $n-1$ applications, anywhere, of this operation. These diagrams represent an infinite subset of the terms in Eq. (4.23). [Examples are shown in Fig. 8(b).] Let us assign the value zero to all other irreducible $C_{2n;p}$. [Examples are shown in
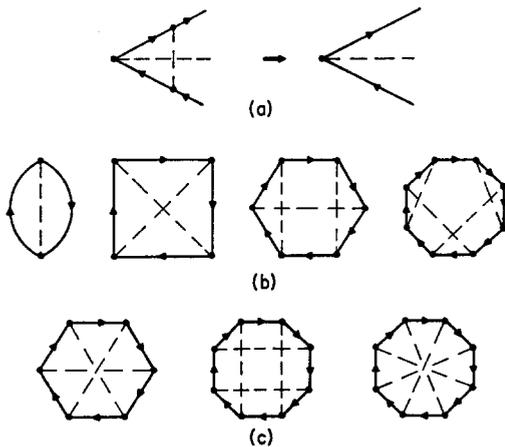
FIG. 8. (a) Contraction operation for the second stochastic model; (b), (c) typical diagrams included and excluded, respectively, in the second stochastic model.

Fig. 8(c).] Now let us take $a=1$. Clearly this implies the value 1, as in the true problem, for all reducible and irreducible $C_{2n;p}$ whose diagrams can be transformed into that for $C_{2;1}$ by repeated application of the line operation of Fig. 4(a) and the vertex operation of Fig. 8(a). All other $C_{2n;p}$ have the value zero.

We have not found an explicit construction for this model of the type provided by Eq. (5.1), et seq., for the random coupling model. Consequently, we have no proof of realizability. As we shall see, however, examination of the dependence of $\tilde{G}(\omega)$ on a, in analogy to Sec. 6, suggests that the model is realizable.

Since the present model retains an infinite subset of terms in Eq. (4.23), it does not directly yield a closed equation for $G(t)$. However, we can obtain a closed system [Eqs. (7.6) and (7.17)] by considering simultaneously the first two equations of an hierarchy analogous to Eq. (2.11). Let

$$H(t) \equiv \sum_\beta \langle H_{\alpha-\beta,\beta,\alpha}(t) \rangle = dG(t)/dt.$$

From Eq. (4.6) we find

$$dH(t)/dt = -\sum_\beta \phi_{\alpha,\beta,\alpha-\beta} \phi_{\alpha-\beta,-\beta,\alpha}$$
$$\times \langle b_\beta b_{-\beta} G_{\alpha,\alpha}(t) \rangle + J(t), \quad H(0)=0, \quad (7.1)$$
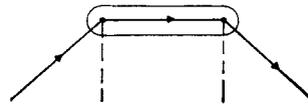
FIG. 9. Fixed diagram part for $J(t)$.

where

$$J(t) = -M^{-1} \sum_{\beta,\gamma}^{(\gamma \neq -\beta)} \phi_{\alpha,\beta,\alpha-\beta} \phi_{\alpha-\beta,\gamma,\alpha-\beta-\gamma}$$
$$\times \langle b_\beta b_\gamma G_{\alpha-\beta-\gamma,\alpha}(t) \rangle. \quad (7.2)$$

Since $G_{\alpha,\alpha}(t)$ is statistically sharp $(M \to \infty)$, it follows from previous relations that the first term on the right-hand side of Eq. (7.1) may be rewritten $-C_{2;1}\langle b^2 \rangle G(t)$. Hence, when $C_{2;1}=1$, we have the equations

$$dG(t)/dt = H(t), \quad G(0)=1,$$
$$dH(t)/dt = -\langle b^2 \rangle G(t) + J(t), \quad H(0)=0. \quad (7.3)$$

By Eq. (4.23) we have

$$H = \sum_{n=1}^{\infty} \sum_p^{irr} (-1)^n C_{2n;p} \langle b^2 \rangle^n G(*G)^{2n-1}. \quad (7.4)$$

An analogous expansion for $J(t)$ may be found by analysis very similar to that which leads to Eq. (4.23). The result is

$$J = \sum_{n=2}^{\infty} \sum_p^{J} (-1)^n C_{2n;p} \langle b^2 \rangle^n G(*G)^{2n-2}, \quad (7.5)$$

where $\sum^J$ is defined as follows: Construct a diagram part as shown in Fig. 9. Call it the fixed part. [The two vertices correspond to the two ϕ factors which appear

explicitly in Eq. (7.2).] The summation $\sum^J$ then is over every closed diagram, constructed by combining two or more vertices with the fixed part, that contains no part, wholly external to the fixed part, which is connected to the rest of the diagram by only solid lines. The three simplest diagrams included are shown in Fig. 10. It will be noted that $\sum^J$ contains both reducible and irreducible diagrams in the sense of Sec. 4. The reducible diagrams all resemble Fig. 10(a) in that the associated C is the product of just two irreducible C's.

Let us denote by Fig. 11(a) the totality of possible diagram parts, with the two solid and single dashed external lines shown, that can be transformed into a single vertex by repeated contractions as shown in Fig. 8(a). Let us call this structure a consolidated vertex part. Then we may represent the entire class of irreducible diagrams which contribute to $H(t)$, in the present model, by the single consolidated diagram shown in Fig. 11(b).[17] In a similar fashion, we may represent by consolidated diagrams all the contributions to $J(t)$ in the present model.[17a] Clearly, two consolidated diagrams included are those shown in Figs. 12(a) and 12(b). If we independently replace the several consolidated vertex parts in these diagrams by all possible
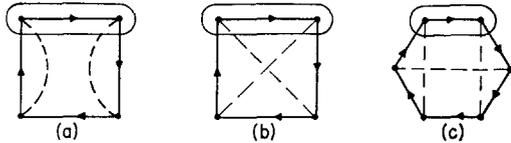
FIG. 10. The simplest diagrams contributing to $J(t)$.

diagram parts which they represent, then we obtain all the individual contributing diagrams which are contractible into Figs. 10(a) or 10(b), respectively. The required contractions do not involve the fixed part. It can be seen that all other diagrams contributing in the present model are represented by the infinite class of consolidated diagrams indicated in Fig. 12(c). These diagrams all may be transformed into that for $C_{2;1}$ by sequences of contractions which now involve the fixed part.

As the representation by consolidated diagrams suggests, the present model leads to a closed expression for $J(t)$ in terms of $H(t)$ and $G(t)$. It is convenient at this point to work with the Laplace transform representation. If $G(p)$, $H(p)$, and $J(p)$ denote the respective transforms of $G(t)$, $H(t)$, and $J(t)$, then the transforms of Eqs. (7.3)–(7.5) are[18]

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[17] It is easy to see that any diagram which is transformable into the diagram for $C_{2;1}$ by contraction operations which involve the fixed vertex may also be so transformed by an alternative sequence of contractions which do not involve this vertex. Thus the diagrams included in this consolidated diagram are exhaustive. (a) Our "consolidated" diagrams are "irreducible" diagrams in the terminology of quantum field theory.

[18] The argument p in $G(p)$, etc., should not be confused with the index p in $C_{2n;p}$.

FIG. 11. (a) Consolidated vertex part; (b) the consolidated diagram for $H(t)$.

$$pG(p)=1+H(p), \quad pH(p)=-\langle b^2\rangle G(p)+J(p), \quad (7.6)$$

$$H(p)=\sum_{n=1}^{\infty}\sum_{p}^{\text{irr}}(-1)^n C_{2n;p}\langle b^2\rangle^n [G(p)]^{2n}, \quad (7.7)$$

$$J(p)=\sum_{n=2}^{\infty}\sum_{p}^{J}(-1)^n C_{2n;p}\langle b^2\rangle^n [G(p)]^{2n-1}. \quad (7.8)$$

Let us write

$$J(p)=J^{(1)}(p)+J^{(2)}(p)+J^{(3)}(p), \quad (7.9)$$

where $J^{(1)}(p)$, $J^{(2)}(p)$, and $J^{(3)}(p)$ are the contributions associated with Figs. 12(a), 12(b), and 12(c), respectively. It can be seen with the aid of the diagrams that the terms of Eq. (7.8) included in $J^{(1)}(p)$ are in one-to-one correspondence to the totality of terms in the expansion

$$[H(p)]^2=\sum_{n,m}\sum_{p,q}^{\text{irr}}(-1)^{n+m}C_{2n;p}C_{2m;q}\langle b^2\rangle^{n+m}$$
$$\times[G(p)]^{2n+2m}, \quad (7.10)$$

which follows from Eq. (7.7). On evaluating the corresponding terms by means of the rule $C_{2n;p}=a^{n-1}$ (all nonvanishing irreducible C's) and then summing, we readily find

$$J^{(1)}(p)=[G(p)]^{-1}[H(p)]^2. \quad (7.11)$$

A similar correspondence exists for $J^{(2)}(p)$, and we thereby find

$$J^{(2)}(p)=a[G(p)]^{-1}[H(p)]^3. \quad (7.12)$$

The evaluation of $J^{(3)}(p)$ is somewhat more involved. Let us write

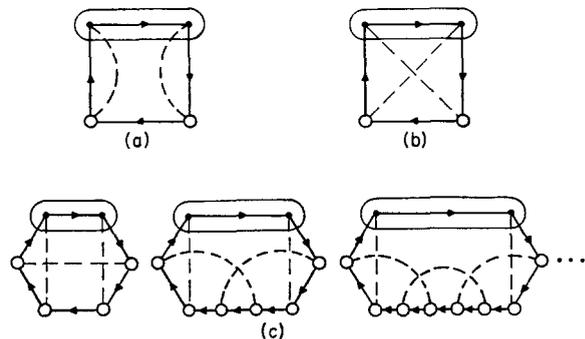$$J^{(3)}(p)=\sum_{i=1}^{\infty}J^{(3,i)}(p), \quad (7.13)$$

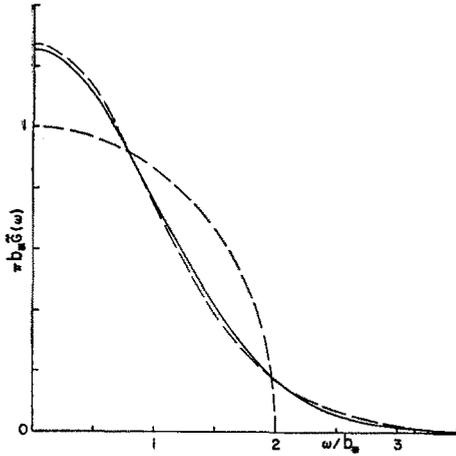FIG. 12. Consolidated diagrams contributing to $J(t)$ in the second stochastic model.

FIG. 13. Comparison of $\tilde{G}(\omega)$ for the random coupling model and the second stochastic model (dashed curves) with $\tilde{G}(\omega)$ for the true problem.

where $J^{(3,i)}(p)$ is the contribution from the ith consolidated diagram in the sequence of Fig. 12(c). The series for $J^{(3,1)}(p)$ is in one-to-one correspondence with that for $[H(p)]^4$, and we find

$$J^{(3,1)}(p) = -a^2\langle b^2\rangle^{-1}[G(p)]^{-3}[H(p)]^4 \quad (7.14)$$

by comparing corresponding terms. In a similar fashion, comparing the series for $J^{(3,i+1)}(p)$ and $[H(p)]^2J^{(3,i)}(p)$, we find

$$J^{(3,i+1)}(p) = -a\langle b^2\rangle^{-1}[G(p)]^{-2}[H(p)]^2J^{(3,i)}(p), \quad (7.15)$$

whence,

$$J^{(3)}(p) = J^{(3,1)}(p) - a\langle b^2\rangle^{-1}[G(p)]^{-2}[H(p)]^2J^{(3)}(p). \quad (7.16)$$

On combining our results (and suppressing the argument p) we have, finally,

$$J = G^{-1}H^2\{1 + [a/(1+a\langle b^2\rangle^{-1}G^{-2}H^2)]\}. \quad (7.17)$$

Eliminating H and J from Eqs. (7.6) and (7.17), we obtain

$$\langle b^2\rangle^2 G^4 + p(\langle b^2\rangle + ap^2)G^3$$
$$- (\langle b^2\rangle + 3ap^2)G^2 + 3apG - a = 0. \quad (7.18)$$

The relevant branch of Eq. (7.18) is the one for which $\tilde{G}(\omega) = \pi^{-1}\text{Re}\{G(-i\omega)\}$ reduces to Eq. (5.6) when $a=0$, and it is easily verified that such a branch exists. Let us examine the behavior of $\tilde{G}(\omega)$ on this branch as a is varied. For sufficiently small a, it is plausible to assume that our assignment of values to the C's is realizable. As we increase a, we may plausibly anticipate that $\tilde{G}(\omega)$ will begin to violate Eq. (2.4), or at least will exhibit some discontinuous change in its dependence on a, when a critical value is reached for which the C's are unrealizable by any assignment of values to the ϕ's consistent with Eq. (4.4). This argument is supported by the example of Sec. 6.

The behavior of Eq. (7.18) is accessible by standard techniques for quartic equations. One finds that the form of $\tilde{G}(\omega)$ on the branch of interest varies continuously with a for $-\frac{1}{4} < a < \infty$. Over this range,

$$\tilde{G}(0) = (\sqrt{2}\pi b_*)^{-1}[1 + (1+4a)^{\frac{1}{2}}]^{\frac{1}{2}}, \quad (7.19)$$

and, for $a>0$, $\tilde{G}(\omega)$ decreases monotonically with increase of ω^2 up to a cutoff frequency given by

$$\omega_c^2 = \frac{1}{2}a^{-1}s^{-2}(1+s)^2[1 - (1-4as)^{\frac{1}{2}}]\langle b^2\rangle, \quad (7.20)$$

where

$$s = 2 - (1+a^{-1}+D^{\frac{1}{2}})^{\frac{1}{2}} - (1+a^{-1}-D^{\frac{1}{2}})^{\frac{1}{2}},$$
$$D = (1+a^{-1})^2 - (1-\tfrac{2}{3}a^{-1})^3.$$

As a increases, ω_c^2 increases monotonically. For $a < -\frac{1}{4}$, $\tilde{G}(0)$ is complex, and we conclude that this gives a lower bound to the range of realizability. There appears to be no upper bound, at least on the basis of the present considerations.

The properties just listed appear to justify the tentative conclusion that the present model is realizable for $a=1$. It should be emphasized, however, that the arguments given do not constitute a proof of realizability. The latter would require an explicit prescription for constructing ϕ's which yield the model. We have not found such a prescription, and, therefore, we regard the investigation of the present model as incomplete.

The function $\tilde{G}(\omega)$ for $a=1$ is compared in Fig. 13 with Eqs. (5.6) and (2.7). It will be noted that the present model gives a close approximation to Eq. (2.7) and represents a substantial improvement over the random coupling model.

The apparent realizability of the present model, and the significant improvement it gives over the random coupling model, suggest that there may be an infinite sequence of closed realizable stochastic models in which successively broader classes of irreducible C's are given the value one and such that $G(t)$ converges rapidly to its value in the true problem. If so, the coupling coefficients $A_{[n,r,s]}$ in Eq. (4.2) may exhibit a distribution, as functions of n, r, and s, which clusters more and more closely about the diagonal values $A_{[n,r,s]} = \delta_{r,n}\delta_{s,n}$ as one ascends the sequence. Thus we may hope that any given dynamical properties of the models should converge in a statistical sense to those of the original collection of uncoupled oscillators. The analytical complexity of models higher than the two already described is formidable.

8. DRIVEN RANDOM OSCILLATOR

Let us add to the right-hand side of Eq. (4.2) a forcing term $f_{[n]}(t)$ which is identically distributed for each n, statistically independent for different n, and statistically independent of $b_{[r]}$ for all n and r. Let us take the initial conditions

$$q_{[n]}(t_0) = 0. \quad (8.1)$$

On writing

$$f_{[n]}(t) = \bar{f}(t) + f_{[n]}'(t), \quad \bar{f}(t) \equiv \langle f_{[n]}(t)\rangle, \quad (8.2)$$

and introducing collective forces $f_\alpha(t)$ defined in correspondence to Eq. (3.1), we find

$$\langle M^{-\frac12} f_0(t)\rangle = \bar{f}(t), \quad \langle |M^{-\frac12} f_0(t) - \bar{f}(t)|^2\rangle = O(M^{-1}),$$
$$\langle f_\alpha(t)\rangle = 0, \quad \langle f_\alpha(t) f_\beta^*(t')\rangle = \delta_{\alpha,\beta} F(t,t') \quad (\alpha \neq 0), \tag{8.3}$$

where

$$F(t,t') = \langle f_{[n]}'(t) f_{[n]}'^*(t')\rangle.$$

We see that $f_0(t)$ plays a special role: When $M \to \infty$, the quantity $M^{-\frac12} f_0(t) = M^{-1}\sum_n f_{[n]}(t)$ takes the sharp value $\bar{f}(t)$. It is easily verified that $f_\alpha(t)$ depends only on the $f_{[n]}'(t)$ if $\alpha \neq 0$.

In place of Eq. (4.1) we now have

$$dq_\alpha(t)/dt = -iM^{-\frac12}\sum_\beta \phi_{\alpha,\beta,\alpha-\beta} b_\beta q_{\alpha-\beta}(t) + f_\alpha(t). \tag{8.4}$$

Suppose that Eq. (8.4) is formally integrated from t_0 and solved by iteration. By arguments similar to those which give Eq. (4.7), we find

$$\langle q_\alpha(t)\rangle = 0 \quad (\alpha \neq 0), \quad \langle q_\alpha(t) q_\gamma^*(t')\rangle = 0 \quad (\alpha \neq \gamma). \tag{8.5}$$

By Eqs. (8.5) and (3.3) we have, for all n,

$$\langle q_{[n]}(t)\rangle = \bar{q}(t), \quad \bar{q}(t) \equiv \langle M^{-\frac12} q_0(t)\rangle. \tag{8.6}$$

It may be verified from a term-by-term examination of the iteration solution that the variance of $M^{-\frac12} q_0(t) = M^{-1}\sum_n q_{[n]}(t)$ is $O(M^{-1})$. Thus, $M^{-\frac12} q_0(t)$ is statistically sharp $(M \to \infty)$ and may be identified with $\bar{q}(t)$.[19]

Because of the special role played by $q_0(t)$, it is convenient to impose, in addition to Eqs. (4.4) and (4.13), the condition[20]

$$\phi_{\mu,\lambda,\sigma} = 1 \quad (\mu \text{ or } \sigma = 0). \tag{8.7}$$

Then from Eq. (8.4), we obtain $(M \to \infty)$

$$d\bar{q}(t)/dt + iM^{-\frac12}\sum_\beta{}' \langle b_{-\beta} q_\beta(t)\rangle = \bar{f}(t), \tag{8.8}$$

$$dq_\alpha(t)/dt = -ib_\alpha \bar{q}(t) - iM^{-\frac12}\sum_\beta{}' \phi_{\alpha,\alpha-\beta,\beta} \\ \times b_{\alpha-\beta} q_\beta(t) + f_\alpha(t) \quad (\alpha \neq 0), \tag{8.9}$$

where $\sum_\beta{}'$ implies that $\beta = 0$ is excluded.

Let us write

$$q_{[n]}(t) = \bar{q}(t) + q_{[n]}'(t), \tag{8.10}$$

where, by Eq. (8.6), $\langle q_{[n]}'(t)\rangle = 0$. We shall call $\bar{q}(t)$ and $q_{[n]}'(t)$ the "coherent" and "incoherent" amplitudes, respectively. An explicit solution for $\bar{q}(t)$ is readily obtained. From the definition of the response matrix, and the statistical independence of the f's and the b's, we have

$$\langle q_0(t)\rangle = \sum_\alpha \int_{t_0}^t \langle G_{0,\alpha}(t-s) f_\alpha(s)\rangle ds$$
$$= \sum_\alpha \int_{t_0}^t \langle G_{0,\alpha}(t-s)\rangle\langle f_\alpha(s)\rangle ds,$$

whence, by Eqs. (8.3), (8.6), and (4.9), we obtain

$$\bar{q}(t) = \int_{t_0}^t G(t-s) \bar{f}(s) ds. \tag{8.11}$$

Consider now the covariance of the incoherent amplitude. Let

$$Q_{[n,m]}(t,t') = q_{[n]}'(t) q_{[m]}'^*(t'), \quad Q_{\alpha,\gamma}(t,t') = q_\alpha(t) q_\gamma^*(t').$$

On using Eqs. (3.3), (8.5), and (8.6), we find

$$\langle Q_{[n,m]}(t,t')\rangle = M^{-1}\sum_\alpha{}' \exp[-i2\pi(n-m)\alpha/M] \\ \times \langle Q_{\alpha,\alpha}(t,t')\rangle + O(M^{-1}). \tag{8.12}$$

Therefore, if $\langle Q_{\alpha,\alpha}(t,t')\rangle$ is independent of α $(\alpha \neq 0)$, we have $(M \to \infty)$

$$\langle Q_{[n,m]}(t,t')\rangle = \delta_{n,m} Q(t,t'), \\ Q(t,t') = \langle Q_{\alpha,\alpha}(t,t')\rangle \quad (\alpha \neq 0). \tag{8.13}$$

Let us assume hereafter that Eq. (8.13) holds. As we shall see shortly, this will be the case when Eqs. (4.13) and (8.7) are satisfied.

An important statistical property is

$$\langle b_\beta b_{-\beta}\cdots b_\mu b_{-\mu} Q_{\alpha,\alpha}(t,t')\rangle - \langle b^2\rangle\cdots\langle b^2\rangle Q(t,t') = O(M^{-1}), \\ (\alpha \neq 0, \ |\alpha| \neq |\beta| \neq \cdots \neq |\mu|). \tag{8.14}$$

For the case of uncoupled oscillators (all ϕ's $= 1$) this follows directly from Eq. (3.1) and the statistical independence of $b_{[n]}, q_{[n]}(t)$ and $b_{[r]}, q_{[r]}(t)$ for $n \neq r$. In the general case, it may be verified for each term of the iteration expansion of the left side of Eq. (8.14).[19]

From Eq. (8.9) we have

$$\partial Q(t,t')/\partial t = S(t,t') + S_C(t,t') + S_F(t,t'), \tag{8.15}$$

where

$$S(t,t') = \sum_\beta{}' \langle S_{\alpha,\alpha-\beta,\beta}(t,t')\rangle \quad (\alpha \neq 0),$$

$$S_{\alpha,\alpha-\beta,\beta}(t,t') = -iM^{-\frac12}\phi_{\alpha,\alpha-\beta,\beta} b_{\alpha-\beta} q_\beta(t) q_\alpha^*(t'), \tag{8.16}$$

$$S_C(t,t') = -i\langle b_\alpha q_\alpha^*(t')\rangle \bar{q}(t) \quad (\alpha \neq 0), \tag{8.17}$$

$$S_F(t,t') = \langle q_\alpha^*(t') f_\alpha(t)\rangle \quad (\alpha \neq 0). \tag{8.18}$$

Our notation anticipates the fact that the expressions given for $S(t,t')$, $S_C(t,t')$, and $S_F(t,t')$ are individually independent of α. It should be noted that the similar equation for $\partial Q(t,t')/\partial t'$ is redundant with Eq. (8.15) because of the property

$$Q(t,t') = Q^*(t',t), \tag{8.19}$$

which follows from the definition of $Q(t,t')$.

The quantity $S_F(t,t')$ is readily evaluated by an argument similar to that which gave Eq. (8.11). We have

$$\langle q_\alpha^*(t') f_\alpha(t)\rangle = \sum_\beta \int_{t_0}^{t'} \langle G_{\alpha,\beta}^*(t'-s)\rangle\langle f_\beta^*(s) f_\alpha(t)\rangle ds,$$

whence

$$S_F(t,t') = \int_{t_0}^{t'} G^*(t'-s) F(t,s) ds. \tag{8.20}$$

The evaluation of $\langle S_{\alpha,\alpha-\beta,\beta}(t,t')\rangle$ parallels that of $\langle H_{\alpha-\beta,\beta,\alpha}(t)\rangle$ in Sec. 4. We expand $q_\beta(t)$ and $q_\alpha^*(t')$ by iteration of the integrated form of Eq. (8.9), leaving

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[19] See Appendix A.

[20] Equation (8.7) constitutes a very weak additional constraint on the ϕ's, and we anticipate that, provided $C_{2;1} = 1$, it will not affect the realizability of any given set of values for the $C_{2n;p}$ when $M \to \infty$. This is verifiably so for the random coupling model: The sums $\sum_\beta C_{2n;p}(\alpha, \beta, \alpha-\beta)$ are unaffected $(M \to \infty)$ for either $\alpha = 0$ or $\alpha \neq 0$.

$\bar{q}(t)$ (which is known) explicitly in the expansion. Then we average and note the regularities imposed by the sum rule for indices and the statistical properties of the b's and the f's. The expansion for $q_\alpha^*(t')$ involves factors ϕ^*. If we express these as ϕ factors by Eq. (4.4), we are led, eventually, to the irreducible diagram expansion

$$\langle S_{\alpha,\alpha-\beta,\beta}(t,t')\rangle$$
$$= M^{-1}\sum_{n=1}^{\infty}\sum_{p}^{\text{irr}} C_{2n;p}(\alpha,\alpha-\beta,\beta)\xi_{2n;p}(t,t') \quad (8.21)$$

$$(\alpha,\beta\neq 0,\ |\alpha|\neq|\beta|\neq|\alpha-\beta|),$$

where the $\xi_{2n;p}(t,t')$ are independent of α and β. From this we obtain[21]

$$S(t,t')=\sum_{n=1}^{\infty}\sum_{p}^{\text{irr}} C_{2n;p}\xi_{2n;p}(t,t'). \quad (8.22)$$

The $\xi_{2n;p}(t,t')$ may be determined by the variational method used for the $\zeta_{2n;p}(t)$ in Sec. 4. For variations which leave the $C_{2n;p}$ unchanged, we have, in correspondence to Eq. (4.16),

$$\Delta\langle S_{\alpha,\alpha-\beta,\beta}(t,t')\rangle$$
$$= M^{-1}\sum_{n}\sum_{p}^{\text{irr}}\xi_{2n;p}(t,t')\Delta C_{2n;p}(\alpha,\alpha-\beta,\beta). \quad (8.23)$$

The variation Eq. (4.17), with the notation change $\beta\to\alpha-\beta$, produces perturbation terms on the right-hand sides of the equations of motion for $q_\alpha^*(t')$ and $q_\beta(t)$. In correspondence to Eq. (4.19), we find (to order $M^{-\frac{1}{2}}$)

$$\Delta q_\alpha^*(t')=\int_{t_0}^{t'} G_{\alpha,\alpha}^*(t'-s)$$
$$\times[iM^{-\frac{1}{2}}\Delta\phi_{\alpha,\alpha-\beta,\beta}^* b_{\beta-\alpha}q_\beta^*(s)]ds,$$
$$\Delta q_\beta(t)=\int_{t_0}^{t} G_{\beta,\beta}(t-s)$$
$$\times[-iM^{-\frac{1}{2}}\Delta\phi_{\beta,\beta-\alpha,\alpha}b_{\beta-\alpha}q_\alpha(s)]ds. \quad (8.24)$$

Then (to order M^{-1}) we find, noting

$$\Delta\phi_{\alpha,\alpha-\beta,\beta}^*=\Delta\phi_{\beta,\beta-\alpha,\alpha},$$

$$\Delta\langle S_{\alpha,\alpha-\beta,\beta}(t,t')\rangle$$
$$= M^{-1}\Delta C_{2;1}(\alpha,\alpha-\beta,\beta)$$
$$\times\left[\int_{t_0}^{t'}\langle G_{\alpha,\alpha}^*(t'-s)b_{\alpha-\beta}b_{\beta-\alpha}q_\beta(t)q_\beta^*(s)\rangle ds\right.$$
$$\left.-\int_{t_0}^{t}\langle G_{\beta,\beta}(t-s)b_{\alpha-\beta}b_{\beta-\alpha}q_\alpha^*(t')q_\alpha(s)\rangle ds\right]. \quad (8.25)$$

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[21] The sums over ϕ products which occur in the present case involve Σ' rather than the unrestricted summation by which the $C_{2n;p}$ are defined in Sec. 4. This does not alter the values of the sums in the limit $M\to\infty$, however.

By using Eq. (8.14) and the sharpness of $G_{\alpha,\alpha}$ and $G_{\beta,\beta}$ to reduce the averages in the limit $M\to\infty$, we have, finally,

$$\xi_{2;1}(t,t')=\langle b^2\rangle\left[\int_{t_0}^{t'} G^*(t'-s)Q(t,s)ds\right.$$
$$\left.-\int_{t_0}^{t} G(t-s)Q^*(t',s)ds\right]. \quad (8.26)$$

It is noteworthy that this expression depends on the driving forces only implicitly, through their effect on $Q(t,t')$.

The higher $\xi_{2n;p}(t,t')$ may be found by introducing more general variations. The result is that $\xi_{2n;p}$ consists of a sum of terms each of which involves a $(2n-1)$-fold time integration over a product of $2n-1$ factors G or G^*, one Q or Q^* factor, and n factors $\langle b^2\rangle$.

We have finally to evaluate $S_C(t,t')$. It can be shown from the iteration solution of Eq. (8.9) that $\langle b_\alpha q_\alpha^*(t')\rangle$, like $\langle Q_{\alpha,\alpha}(t,t')\rangle$, has an irreducible diagram expansion and is independent of α ($\alpha\neq 0$). From the latter fact we have

$$\langle b_\alpha q_\alpha^*(t')\rangle=M^{-1}\sum_{\beta}'\langle b_\beta q_\beta^*(t')\rangle+O(M^{-1}).$$

Hence, in the limit $M\to\infty$ we obtain from Eq. (8.8) the result

$$S_C(t,t')=-\bar{q}(t)[d\bar{q}^*(t')/dt'-\bar{f}^*(t')]. \quad (8.27)$$

Let us now specialize to the random coupling model. By Eq. (5.3), we then have[20]

$$S(t,t')=\xi_{2;1}(t,t'). \quad (8.28)$$

Equations (8.15), (8.19), (8.20), (8.26), (8.27), and (8.28) now permit the determination of $Q(t,t')$.

Twice the real part of Eq. (8.15) for $t=t'$ is the equation for the rate of change of the mean intensity $Q(t,t)$ of the incoherent oscillation. The quantities

$$2\operatorname{Re}\{S_C(t,t)\}\quad\text{and}\quad 2\operatorname{Re}\{S_F(t,t)\}$$

represent contributions to $dQ(t,t)/dt$ from interaction with the coherent oscillation and incoherent driving forces respectively. From Eqs. (8.26) and (8.28) we have

$$\operatorname{Re}\{S(t,t)\}=0. \quad (8.29)$$

Thus, using Eq. (8.27), we verify the conservation of total intensity,

$$(d/dt)[Q(t,t)+\bar{q}(t)\bar{q}^*(t)]=0, \quad (8.30)$$

when all the driving forces vanish. This consistency property, and Eq. (8.29) itself, are assured in advance because our equations constitute the exact description of a model for which Eq. (4.4) holds. It is also assured that the solution $Q(t,t')$ of our model equations obeys all the realizability conditions to which covariances are subject.[22]

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[22] One such condition is $|Q(t,t')|^2\leq Q(t,t)Q(t',t')$.

The simplest solution of Eq. (8.15) for the random coupling model results from taking forces which vanish except for impulses at $t=t_0$ such that $Q(t_0,t_0)=1$, $\dot{q}(t_0)=0$. Then it is easily seen from Eq. (8.26), and the property $G(t)=G^*(-t)$,[23] that Eq. (8.15) becomes identical with Eq. (5.4) (for t, $t'>t_0$) under the substitution

$$Q(t,t')=G(t-t'). \qquad (8.31)$$

Thus Eq. (8.31) is the solution for random shock excitation, as is required for consistency. In general, $Q(t,t')$ and $G(t-t')$ do not have the same form.

9. NON-GAUSSIAN FREQUENCY DISTRIBUTION

We shall now describe briefly the generalizations required when the distribution of $b_{[n]}$ is non-Gaussian. The case $\langle b_{[n]}\rangle\neq0$ can be treated by methods similar to those used for $\langle f_{[n]}\rangle\neq0$ in Sec. 8. But it is simpler to maintain the condition $\langle b_{[n]}\rangle=0$ and instead replace Eq. (8.4) by

$$(d/dt+i\bar{\omega}+\nu)q_\alpha(t)$$
$$+iM^{-\frac{1}{2}}\sum_\beta \phi_{\alpha,\beta,\alpha-\beta}b_\beta q_{\alpha-\beta}(t)=f_\alpha(t), \quad (9.1)$$

where $\bar{\omega}$ is the (real) mean frequency and we have also included a real damping factor ν. It is easy to see that this generalization implies only minor changes in our treatment if $b_{[n]}$ remains Gaussian. The factor $t^{2n}/2n!$ in the iteration series Eq. (4.14) is replaced by $G^{(0)}(*G^{(0)})^{2n}$, where

$$G^{(0)}(t)=\exp[-(i\bar{\omega}+\nu)t]. \qquad (9.2)$$

However, the irreducible diagram expansions for $H(t)$ and $S(t,t')$, and the expressions for $\zeta_{2n;p}(t)$ and $\xi_{2n;p}(t,t')$, are unchanged in form. The effect of $\bar{\omega}$, $\nu\neq0$ is implicitly expressed by the changed *values* of the functions G and Q which appear in these expressions. The only further changes in Secs. 2–8 are the obvious replacements

$$d/dt \rightarrow (d/dt+i\bar{\omega}+\nu) \quad \text{and} \quad \partial/\partial t \rightarrow (\partial/\partial t+i\bar{\omega}+\nu)$$

where appropriate.

Now let us consider the general (non-Gaussian) case where the $b_{[n]}$ are identically distributed (with zero mean) for all n and statistically independent for different n. It is easy to verify from Eqs. (3.1) and (3.2) that Eq. (3.8) remains valid. In place of Eq. (3.10), we find

$$\langle b_\alpha\rangle=0, \quad \langle b_\alpha b_\beta\rangle=\delta_{\alpha+\beta}\langle b^2\rangle,$$

$$\langle b_\alpha b_\beta b_\gamma\rangle=\delta_{\alpha+\beta+\gamma}M^{-\frac{1}{2}}\langle b^3\rangle,$$

$$\langle b_\alpha b_\beta b_\gamma b_\epsilon\rangle=(\delta_{\alpha+\beta}\delta_{\gamma+\epsilon}+\delta_{\alpha+\gamma}\delta_{\beta+\epsilon}+\delta_{\alpha+\epsilon}\delta_{\beta+\gamma})\langle b^2\rangle^2 \qquad (9.3)$$
$$+\delta_{\alpha+\beta+\gamma+\epsilon}M^{-1}(\langle b^4\rangle-3\langle b^2\rangle^2),$$

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In general we find $(M\rightarrow\infty)$ that all moments of the b_α with indices equal and opposite in pairs depend only

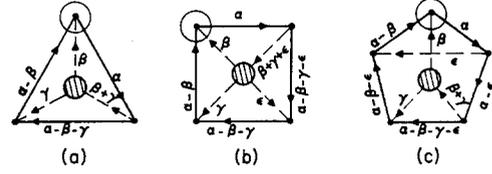[23] This property of Eq. (5.5) is directly implied by Eq. (2.4).

FIG. 14. Diagrams for simple non-Gaussian contributions to $G(t)$.

on $\langle b^2\rangle$ and have the same values as for a Gaussian distribution of $b_{[n]}$ with this variance. Moments for which the indices do not pair (we shall call them skew moments) have values which depend on the cumulants of the $b_{[n]}$ distribution; they tend individually to zero as $M\rightarrow\infty$.[24]

The presence of skew moments results in new classes of terms in Eq. (4.11) and, consequently, in Eq. (4.23). The simplest new term in Eq. (4.23) is

$$(-i)^3 D_{3;1}\langle b^3\rangle G*G*G, \qquad (9.4)$$

where

$$D_{3;1}=M^{-2}\sum_{\beta,\gamma}\phi_{\alpha,\beta,\alpha-\beta}\phi_{\alpha-\beta,\gamma,\alpha-\beta-\gamma}\phi_{\alpha-\beta-\gamma,-\beta-\gamma,\alpha}. \quad (9.5)$$

It may be represented by Fig. 14(a). The further terms represented by Figs. 14(b) and 14(c) are proportional to $\langle b^4\rangle-3\langle b^2\rangle^2$ and to $\langle b^2\rangle\langle b^3\rangle$, respectively.

It is possible to generalize our sequence of models so that closed equations are produced which systematically include more and more of the information expressed by the cumulants of the b distribution. We shall not attempt this here. However, it is important to note that the equations for the random coupling model are identical to those already given no matter what the (zero-mean) b distribution may be. It is clear that $D_{3;1}$ vanishes for this model $(M\rightarrow\infty)$, and it can be seen that all the higher new terms in Eq. (4.23) vanish also. The statistical properties of the random coupling model thus depend only on the variance $\langle b^2\rangle$. On recalling Eq. (2.3), which is exact for the original uncoupled oscillators, we see that in certain respects the random coupling model actually will provide a better approximation for distributions of b which resemble Eq. (5.6) in form than for a Gaussian distribution.

In the physical analogs to the random oscillator which are our eventual interest, the distribution of the stochastic quantity corresponding to b may itself be determined by dynamical processes. In this case there may exist an alternative to the generalized treatment we have mentioned. It may be physically reasonable to assume Gaussian *initial conditions* for the quantities corresponding to b and q. If the dynamical equations for these quantities are then treated as a simultaneous set, the non-Gaussian diagrams will not arise in any of the relevant sequence of models. We shall give an illustration at the end of Sec. 11.

[24] However, their number increases correspondingly so that those sums over individual skew moments which contribute to physical quantities remain finite in the limit.

10. PARTICLE IN A RANDOM POTENTIAL

Let the Schrödinger equation for a particle be[25]

$$(\partial/\partial t - i\nabla^2)\psi(\mathbf{x},t) = -iv(\mathbf{x})\psi(\mathbf{x},t), \qquad (10.1)$$

where $v(\mathbf{x})$ is a real potential which is statistically distributed over an infinite ensemble of realizations of the system. This problem is an exact homolog to the random oscillator with respect to treatment by stochastic models.[26] Let us consider a collection of M systems such that the individual potentials $v_{[n]}(\mathbf{x})$ are identically distributed for all n and are statistically independent for different n. Let $\psi_{[n]}(\mathbf{x},t)$ be the Schrödinger function for the nth system. Then we may define the collective quantities $\psi_\alpha(\mathbf{x},t)$ and $v_\alpha(\mathbf{x})$ in correspondence to Eq. (3.1), and consider the model equations

$$(\partial/\partial t - i\nabla^2)\psi_\alpha(\mathbf{x},t) = -iM^{-\frac{1}{2}}\sum_\beta \phi_{\alpha,\beta,\alpha-\beta}$$
$$\times v_\beta(\mathbf{x})\psi_{\alpha-\beta}(\mathbf{x},t). \quad (10.2)$$

The $\phi_{\alpha,\beta,\alpha-\beta}$ will be identical quantities for corresponding models in the present problem and the random oscillator problem.

The condition Eq. (4.4) serves to maintain hermiticity in the present case. It is easily verified from Eqs. (4.4) and (10.2) that the total probability,

$$\sum_n \int |\psi_{[n]}(\mathbf{x},t)|^2 d^3x = \sum_\alpha \int |\psi_\alpha(\mathbf{x},t)|^2 d^3x$$

and the total energy,

$$\int \left[-\sum_n \psi_{[n]}{}^*\nabla^2\psi_{[n]} + \sum_{n,r,s} \psi_{[n]}{}^*A_{[n,r,s]}v_{[r]}\psi_{[s]} \right] d^3x,$$

where $A_{[n,r,s]}$ is defined by Eq. (4.3), are conserved. However, the individual quantities $\int |\psi_{[n]}|^2 d^3x$ are not constants of motion, in general. The systems in the collection exchange particles as well as energy.

Let us now take the case where $v_{[n]}(\mathbf{x})$ has a multivariate Gaussian distribution. This implies that all odd-order moments vanish and that all even-order moments are expressible in terms of the covariance $V(\mathbf{x},\mathbf{x}') = \langle v_{[n]}(\mathbf{x})v_{[n]}(\mathbf{x}') \rangle$. In the collective representation we have

$$\langle v_\alpha(\mathbf{x})v_\beta(\mathbf{x}') \rangle = \delta_{\alpha+\beta} V(\mathbf{x},\mathbf{x}'),$$

$$\langle v_\alpha(\mathbf{x})v_\beta(\mathbf{x}')v_\gamma(\mathbf{x}'')v_\epsilon(\mathbf{x}''') \rangle$$
$$= \delta_{\alpha+\beta}\delta_{\gamma+\epsilon} V(\mathbf{x},\mathbf{x}') V(\mathbf{x}'',\mathbf{x}''')$$
$$+ \delta_{\alpha+\gamma}\delta_{\beta+\epsilon} V(\mathbf{x},\mathbf{x}'') V(\mathbf{x}',\mathbf{x}''') \qquad (10.3)$$
$$+ \delta_{\alpha+\epsilon}\delta_{\beta+\gamma} V(\mathbf{x},\mathbf{x}''') V(\mathbf{x}',\mathbf{x}''),$$

$$\cdots\cdots$$

The analog of Eq. (3.8) holds, of course, whatever the distribution.

Let us define the Green's function $G_{[n,m]}(\mathbf{x},t|\mathbf{x}',t')$ as the solution (for all t) of the model equation for $\psi_{[n]}(\mathbf{x},t)$ under the initial condition

$$\psi_{[r]}(\mathbf{x},t') = \delta_{r,m}\delta(\mathbf{x}-\mathbf{x}') \quad \text{(all } r\text{)},$$

and make a corresponding definition for $G_{\alpha,\gamma}(\mathbf{x},t|\mathbf{x}',t')$. Then, in correspondence to the analysis in Sec. 4, we find, when Eq. (4.13) is satisfied,

$$(\partial/\partial t - i\nabla_x{}^2)G(\mathbf{x},t|\mathbf{x}',t')$$

$$= \sum_{n=1}^\infty \sum_p^{irr} (-1)^n C_{2n;p}\zeta_{2n;p}(\mathbf{x},t|\mathbf{x}',t'), \quad (10.4)$$

$$G(\mathbf{x},t'|\mathbf{x}',t') = \delta(\mathbf{x}-\mathbf{x}'),$$

where

$$\langle G_{[n,m]}(\mathbf{x},t|\mathbf{x}',t') \rangle = \delta_{n,m}G(\mathbf{x},t|\mathbf{x}',t'),$$
$$\langle G_{\alpha,\gamma}(\mathbf{x},t|\mathbf{x}',t') \rangle = \delta_{\alpha,\gamma}G(\mathbf{x},t|\mathbf{x}',t'), \qquad (10.5)$$

and $\zeta_{2n;p}(\mathbf{x},t|\mathbf{x}',t')$ is the homolog of $\zeta_{2n;p}(t-t')$.

The functions $\zeta_{2n;p}(\mathbf{x},t|\mathbf{x}',t')$ may be determined by the variational method used in Sec. 4. The variation Eq. (4.17) produces, in correspondence to Eq. (4.19), the variation

$$\Delta G_{\alpha-\beta,\alpha}(\mathbf{x},t|\mathbf{x}',t')$$

$$= \int_{t'}^t ds \int d^3y\, G_{\alpha-\beta,\alpha-\beta}(\mathbf{x},t|\mathbf{y},s)$$

$$\times [-iM^{-\frac{1}{2}}\Delta\phi_{\alpha-\beta,-\beta,\alpha}v_{-\beta}(\mathbf{y})G_{\alpha,\alpha}(\mathbf{y},s|\mathbf{x}',t')], \quad (10.6)$$

and we are led, thereby, to the result

$$\zeta_{2;1}(\mathbf{x},t|\mathbf{x}',t')$$

$$= \int_{t'}^t ds \int d^3y\, V(\mathbf{x},\mathbf{y})G(\mathbf{x},t|\mathbf{y},s)G(\mathbf{y},s|\mathbf{x}',t'), \quad (10.7)$$

which corresponds to Eq. (4.21). This result depends on the fact that $G_{\alpha,\alpha}(\mathbf{x},t|\mathbf{x}',t')$ is statistically sharp $(M \to \infty)$, which may be demonstrated in the same way as for $G_{\alpha,\alpha}(t)$ of Sec. 4.

The result for $\zeta_{4;3}(\mathbf{x},t|\mathbf{x}',t')$ is

$$\zeta_{4;3}(\mathbf{x},t|\mathbf{x}',t')$$

$$= \int_{t'}^t ds_1 \int_{t'}^{s_1} ds_2 \int_{t'}^{s_2} ds_3 \int\int\int d^3y_1 d^3y_2 d^3y_3$$

$$\times V(\mathbf{x},\mathbf{y}_2) V(\mathbf{y}_1,\mathbf{y}_3)G(\mathbf{x},t|\mathbf{y}_1,s_1)G(\mathbf{y}_1,s_1|\mathbf{y}_2,s_2)$$

$$\times G(\mathbf{y}_2,s_2|\mathbf{y}_3,s_3)G(\mathbf{y}_3,s_3|\mathbf{x}',t'). \quad (10.8)$$

The structure of expressions (10.7) and (10.8) may be represented, as in Fig. 15, by an appropriate labeling of the vertices in the diagrams for $C_{2;1}$ and $C_{4;3}$. The expressions for all the higher $\zeta_{2n;p}(\mathbf{x},t|\mathbf{x}',t')$ may be written down by analogy from the diagrams for the corresponding irreducible $C_{2n;p}$.

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[25] We take units such that $\hbar = 1$ and $2m = 1$, where m is the particle mass.

[26] The two problems may be regarded as formally identical if b and q are interpreted as vectors in a function space and a correspondence is established between d/dt and $\partial/\partial t - i\nabla^2$.

In the random coupling model, where $C_{2;1}=1$ and all the higher irreducible $C_{2n;p}$ vanish, Eq. (10.4) becomes

$$(\partial/\partial t - i\nabla_x^2)G(\mathbf{x},t\,|\,\mathbf{x}',t')$$

$$= -\int_{t'}^t ds \int d^3y\, V(\mathbf{x},\mathbf{y})G(\mathbf{x},t\,|\,\mathbf{y},s)G(\mathbf{y},s\,|\,\mathbf{x}',t'), \quad (10.9)$$

$$G(\mathbf{x},t'\,|\,\mathbf{x}',t')=\delta(\mathbf{x}-\mathbf{x}').$$

It should be pointed out that this result is independent of the assumption that the potential has a Gaussian distribution, provided that $\langle v_{[n]}(\mathbf{x})\rangle=0$ (cf. Sec. 9).[27]

We are assured that the solutions of Eq. (10.9) will exhibit certain consistency properties because this is an exact equation for a realizable model. In particular, if Eq. (10.2) is transformed into the momentum representation, it follows from a straightforward extension of the arguments given in connection with Eq. (5.7) that $G(\mathbf{x},t\,|\,\mathbf{x}',t')$ satisfies a basic spectral condition. The latter takes its simplest form for the homogeneous case $V(\mathbf{x},\mathbf{x}')=V(\mathbf{x}-\mathbf{x}')$, in which $G(\mathbf{x},t\,|\,\mathbf{x}',t')$ can depend only on $\mathbf{x}-\mathbf{x}'$ and $t-t'$. If we write

$$G_{\mathbf{k}}(t-t')=\int d^3y\, G(\mathbf{x},t\,|\,\mathbf{x}',t')\exp(-i\mathbf{k}\cdot\mathbf{y})$$

$$(\mathbf{y}=\mathbf{x}-\mathbf{x}'), \quad (10.10)$$

$$\tilde{G}_{\mathbf{k}}(\omega)=(2\pi)^{-1}\int_{-\infty}^{\infty}ds\, G_{\mathbf{k}}(s)\exp(i\omega s),$$

then the spectral condition is

$$\tilde{G}_{\mathbf{k}}(\omega)=|\tilde{G}_{\mathbf{k}}(\omega)|. \quad (10.11)$$

[We may note that $G_{\mathbf{k}}(t-t')$ is the diagonal response function for the amplitude in the mode $\mathbf{k}$.] Equation (10.11) implies the reciprocity relation

$$G(\mathbf{x},t\,|\,\mathbf{x}',t')=G^*(\mathbf{x}',t'\,|\,\mathbf{x},t). \quad (10.12)$$

When $V(\mathbf{x},\mathbf{x}')=V(\mathbf{x}-\mathbf{x}')$, Eq. (10.9) has the transform

$$(\partial/\partial t + ik^2)G_{\mathbf{k}}(t)=-\int^t ds \int d^3k'\, V_{\mathbf{k}-\mathbf{k}'}G_{\mathbf{k}'}(t-s)G_{\mathbf{k}}(s),$$

$$G_{\mathbf{k}}(0)=1, \quad (10.13)$$

where

$$V_{\mathbf{k}}=(2\pi)^{-3}\int d^3y\, V(\mathbf{y})\exp(-i\mathbf{k}\cdot\mathbf{y}).$$

It is possible to solve Eq. (10.13) easily for very high k (the WKBJ limit). This is of particular interest because it is well known that the perturbation approach breaks down in this limit. Let us take k sufficiently high that $V_{\mathbf{k}-\mathbf{k}'}\approx 0$, unless $|\mathbf{k}-\mathbf{k}'|\ll k$. Then it is plausible that we may replace $G_{\mathbf{k}'}(t-s)$ by $G_{\mathbf{k}}(t-s)$ in Eq. (10.13) and

FIG. 15. Diagrams for $\zeta_{2;1}(\mathbf{x},t\,|\,\mathbf{x}',t')$ and $\zeta_{4;3}(\mathbf{x},t\,|\,\mathbf{x}',t')$.

thereby obtain[28]

$$(\partial/\partial t + ik^2)G_{\mathbf{k}}(t)=-\langle v^2\rangle\int_0^t G_{\mathbf{k}}(t-s)G_{\mathbf{k}}(s)ds,$$

$$G_{\mathbf{k}}(0)=1, \quad (10.14)$$

where

$$\langle v^2\rangle = V(0)=\int V_{\mathbf{k}}d^3k.$$

Equation (10.14) becomes identical with Eq. (5.4) under the transformation

$$G_{\mathbf{k}}(t)\rightarrow \exp(-ik^2t)G(t), \quad \langle v^2\rangle \rightarrow \langle b^2\rangle.$$

Consequently, we have

$$\tilde{G}_{\mathbf{k}}(\omega)=(\pi v_*)^{-1}\left[1-\left(\frac{\omega-k^2}{2v_*}\right)^2\right]^{\frac{1}{2}} \quad (|\omega-k^2|\leq 2v_*),$$

$$=0 \quad (|\omega-k^2|> 2v_*), \quad (10.15)$$

where $v_*=\langle v^2\rangle^{\frac{1}{2}}$. This correspondence between the WKBJ limit and the random oscillator is not confined to the random coupling model. If Eq. (10.4) is written in the k representation in this limit, and the previous transformation is made, the resulting equation is identical with Eq. (4.23). A particular consequence is that the WKBJ solution to the true problem (all ϕ's=1) is

$$\tilde{G}_{\mathbf{k}}(\omega)=(2\pi\langle v^2\rangle)^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(\omega-k^2)^2/\langle v^2\rangle], \quad (10.16)$$

in correspondence to Eq. (2.7). This result states that for sharp kinetic energy (sharp k) the total energy distribution follows the Gaussian potential energy distribution. Considered as an approximation to Eq. (10.16), the random coupling result (10.15) exhibits the qualitative physical fact that sharp momentum states are not sharp energy states. The quantitative form of Eq. (10.15) suggests that the random coupling model may represent a better approximation to the true problem if the true potential distribution has a clipped rather than a Gaussian form (cf. Sec. 9). It should be noted that the cumulant-discard approximation scheme, when applied in the WKBJ limit, yields expressions for $\exp(ik^2t)G_{\mathbf{k}}(t)$ which are identical in form to Eq. (2.13). This implies discrete spectra $\tilde{G}_{\mathbf{k}}(\omega)$, which is unphysical compared to the random coupling result.

The general correspondence between the WKBJ limit and the random oscillator includes, of course, the second stochastic model, discussed in Sec. 7. The WKBJ

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[27] When the latter condition is not fulfilled, an additional term $i\theta(\mathbf{x})G(\mathbf{x},t\,|\,\mathbf{x}',t')$ appears on the left-hand side of Eq. (10.4), where $\theta(\mathbf{x})$ is the mean potential, and $V(\mathbf{x},\mathbf{y})$ then is defined in terms of the zero-mean part of the potential.
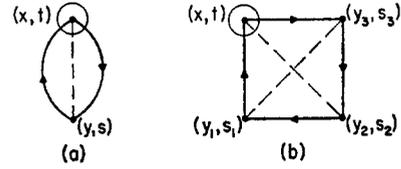
[28] This procedure can be justified a posteriori.

results for the two stochastic models and for the true problem in the Gaussian case are given by Fig. 13, if the horizontal and vertical axes are relabeled $(\omega - k^2)/v_*$ and $\pi v_* \bar{G}_k(\omega)$, respectively. Away from the WKBJ limit, the analysis of the second model is considerably more difficult than for the random oscillator, although the same in principle. The equations are not reducible to algebraic form, and the analogs to the inverses $[G(p)]^{-1}$ and $\langle b^2 \rangle^{-1}$, which appeared in Sec. 7, must be defined by integral equations.

Let us assume that the Schrödinger fields are switched on at $t = t_0$ in such a way that the $\psi_{[n]}(x,t_0)$ are statistically independent and identically distributed for all n and statistically independent of $v_{[r]}(x)$ for all r. Let

$$\psi_{[n]}(\mathbf{x},t) = \bar{\psi}(\mathbf{x},t) + \psi_{[n]}'(\mathbf{x},t),$$
$$\bar{\psi}(\mathbf{x},t) = \langle \psi_{[n]}(\mathbf{x},t) \rangle. \tag{10.17}$$

We shall call $\bar{\psi}(\mathbf{x},t)$ the coherent wave and $\psi_{[n]}'(\mathbf{x},t)$ the incoherent wave. The evolution of the coherent amplitude and the incoherent covariance may be determined by direct correspondence to the analysis in Sec. 8.

Noting that our switch-on is equivalent to the action of impulsive sources $f_{[n]}(\mathbf{x},t) = \psi_{[n]}(\mathbf{x},t_0)\delta(t-t_0)$, we have, in correspondence to Eq. (8.11),

$$\bar{\psi}(\mathbf{x},t) = \int G(\mathbf{x},t \mid \mathbf{y},t_0)\bar{\psi}(\mathbf{y},t_0)d^3y. \tag{10.18}$$

When the potential is statistically homogeneous, Eq. (10.18) has the transform

$$\bar{\psi}_k(t) = G_k(t - t_0)\bar{\psi}_k(t_0), \tag{10.19}$$

where

$$\bar{\psi}_k(t) = (2\pi)^{-3}\int \bar{\psi}(\mathbf{x},t)\exp(-i\mathbf{k}\cdot\mathbf{x})d^3x.$$

In this case the various momentum modes of the coherent wave evolve independently. As our WKBJ limit results illustrate, $G_k(t)$ has a continuous spectrum and, therefore, vanishes as $t \to \infty$. Consequently, the coherent wave eventually is extinguished by its interaction with the random potential.

In direct correspondence to the results obtained in Sec. 8, we have

$$\langle \psi_{[n]}'(\mathbf{x},t)\psi_{[m]}'^*(\mathbf{x}',t') \rangle = \delta_{n,m}\Psi(\mathbf{x},t; \mathbf{x}',t'),$$
$$\langle \psi_\alpha(\mathbf{x},t)\psi_\gamma^*(\mathbf{x}',t') \rangle = \delta_{\alpha,\gamma}\Psi(\mathbf{x},t; \mathbf{x}',t') \tag{10.20}$$
$$(\alpha \neq 0).$$

where $\Psi(\mathbf{x},t; \mathbf{x}',t')$ has the symmetry property

$$\Psi(\mathbf{x},t; \mathbf{x}',t') = \Psi^*(\mathbf{x}',t'; \mathbf{x},t), \tag{10.21}$$

and obeys

$$(\partial/\partial t - i\nabla_x^2)\Psi(\mathbf{x},t; \mathbf{x}',t')$$
$$= S(\mathbf{x},t; \mathbf{x}',t') + S_C(\mathbf{x},t; \mathbf{x}',t'), \tag{10.22}$$

with

$$S(\mathbf{x},t; \mathbf{x}',t') = \sum_{n=1}^{n}\sum_{p}^{irr} C_{2n;p}\xi_{2n;p}(\mathbf{x},t; \mathbf{x}',t'), \tag{10.23}$$

and

$$S_C(\mathbf{x},t; \mathbf{x}',t') = -\bar{\psi}(\mathbf{x},t)(\partial/\partial t' + i\nabla_{x'}^2)\bar{\psi}^*(\mathbf{x}',t'). \tag{10.24}$$

There is no term corresponding to $S_F(t,t')$ because we have not admitted sources for $t > t_0$.

The functions $\xi_{2n;p}(\mathbf{x},t; \mathbf{x}',t')$, which are homologous to the $\xi_{2n;p}(t,t')$ in Eq. (8.22), may be determined by employing our variational procedure and noting the statistical property

$$\langle v_\beta(\mathbf{x}_1)v_{-\beta}(\mathbf{x}_1') \cdots v_\mu(\mathbf{x}_r)v_{-\mu}(\mathbf{x}_r')\psi_\alpha(\mathbf{y},t)\psi_\alpha^*(\mathbf{y}',t') \rangle$$
$$= V(\mathbf{x}_1,\mathbf{x}_1') \cdots V(\mathbf{x}_r,\mathbf{x}_r')\Psi(\mathbf{y},t; \mathbf{y}',t') + O(M^{-1}) \tag{10.25}$$
$$(\alpha \neq 0, \quad |\alpha| = |\beta| \neq \cdots \neq |\mu|),$$

which corresponds to Eq. (8.14). (Here $\mathbf{x}_1$, $\mathbf{x}_1'$, etc., are arbitrary position vectors.) In particular, we find

$$\xi_{2;1}(\mathbf{x},t; \mathbf{x}',t')$$
$$= \int d^3y V(\mathbf{x},\mathbf{y})\left[\int_{t_0}^{t'} ds G^*(\mathbf{x}',t' \mid \mathbf{y},s)\Psi(\mathbf{x},t; \mathbf{y},s)\right.$$
$$\left. - \int_{t_0}^{t} ds G(\mathbf{x},t \mid \mathbf{y},s)\Psi^*(\mathbf{x}',t'; \mathbf{y},s)\right]. \tag{10.26}$$

For the random coupling model,

$$S(\mathbf{x},t; \mathbf{x}',t') = \xi_{2;1}(\mathbf{x},t; \mathbf{x}',t'), \tag{10.27}$$

and we have a closed set of equations which determine $\Psi(\mathbf{x},t; \mathbf{x}',t')$ when the initial functions $\Psi(\mathbf{x},t_0; \mathbf{x}',t_0)$ and $\bar{\psi}(\mathbf{x},t_0)$ are given. As was the case for $G(\mathbf{x},t \mid \mathbf{x}',t')$, certain important consistency properties necessarily are exhibited by the solution $\Psi(\mathbf{x},t; \mathbf{x}',t')$ for any realizable model. In particular, we are assured that $\Psi(\mathbf{x},t; \mathbf{x}',t')$ satisfies all the realizability conditions to which a covariance is subject. In the homogeneous case, where the spatial dependence of $\Psi(\mathbf{x},t; \mathbf{x}',t')$ involves only $\mathbf{x} - \mathbf{x}'$, we must have

$$\Psi_k(t,t) = |\Psi_k(t,t)|, \tag{10.28}$$

where

$$\Psi_k(t,t') = (2\pi)^{-3}\int \Psi(\mathbf{x},t; \mathbf{x}',t')\exp(-i\mathbf{k}\cdot\mathbf{y})d^3y$$
$$(\mathbf{y} = \mathbf{x} - \mathbf{x}').$$

When $t_0 \to -\infty$ and a statistically stationary state has been set up, so that $\Psi_k(t,t') = \Psi_k(t-t')$, we have, further,

$$\tilde{\Psi}_k(\omega) = |\tilde{\Psi}_k(\omega)|, \tag{10.29}$$

where

$$\tilde{\Psi}_k(\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} \Psi_k(t)\exp(i\omega t)dt.$$

In contrast, cumulant-discard approximations similar to those of Sec. 2 may lead, in the present problem, to *negative* occupation probabilities, $\Psi_k(t,t) < 0$, for physically admissible initial conditions. Such behavior is easily verified in simpler, but analogous, dynamical systems.

Twice the real part of Eq. (10.22) for $x',t'=x,t$ represents the continuity equation for the ensemble mean of the quantum-mechanical probability of finding a particle. The left-hand side is the quantum-mechanical equivalent of the substantial derivative of the mean probability density $\Psi(x,t; x,t)$ in the incoherent wave. The corresponding quantity for the coherent wave is $-2 \text{Re}\{S_C(x,t; x,t)\}$. It is clear from Eqs. (10.27) and (10.26) that $\text{Re}\{S(x,t; x,t)\}$ vanishes. Consequently, the continuity equation simply states that a particle enters the incoherent wave as it leaves the coherent wave.

The vanishing of $\text{Re}\{S(x,t; x,t)\}$ expresses the fact that the direct effect of the potential on the particles is to change their momentum rather than their position. To illustrate this, let us take $\bar{\psi}(x,t)=0$ and assume that the potential and the incoherent wave are statistically homogeneous. Then from the Fourier transforms of Eqs. (10.22), (10.27), and (10.26), we obtain

$$d\Psi_k(t,t)/dt$$

$$= 2 \text{Re} \int_{t_0}^t ds \int d^3k' V_{k-k'} [G_k(s-t)\Psi_{k'}(t,s) - G_{k'}(t-s)\Psi_k(s,t)], \quad (10.30)$$

after noting Eqs. (10.12) and (10.21). The quantity $\Psi_k(t,t)$ is the mean probability density for finding a particle with momentum k. The right side of Eq. (10.30), therefore, is the rate of transfer of particles to this momentum from all other momenta k'. It is easily verified from Eq. (10.30) that $\int \Psi_k(t,t)d^3k$ is a constant of motion.

Suppose that the fields have been switched on at $t_0=-\infty$ in such fashion that a stationary state exists at time t. By using Eqs. (10.11) and (10.29), the right-hand side of Eq. (10.30) may be rewritten so that we have

$$0 = d\Psi_k(t,t)/dt$$

$$= \int_{-\infty}^{\infty} d\omega \int d^3k' V_{k-k'} [\tilde{G}_k(\omega)\tilde{\Psi}_{k'}(\omega) - \tilde{G}_{k'}(\omega)\tilde{\Psi}_k(\omega)]. \quad (10.31)$$

We note that the right-hand side of Eq. (10.31) is the difference of two terms each of which is positive.[29] The first represents an input of particles to mode k from other modes k' and the second represents an output to these other modes. If the excitation of mode k only were to be slowly increased by some outside agency, it is clear that the output term would increase in magnitude while the input term would be initially unaffected. Thus, the random coupling model exhibits a plausible tendency to restore statistical equilibrium.

It will be noted that Eq. (10.31) is satisfied in general

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if

$$\tilde{\Psi}_k(\omega) = f(\omega)\tilde{G}_k(\omega), \quad (10.32)$$

where $f(\omega)$ is a function independent of k. Now it can be seen from their definitions that $\tilde{G}_k(\omega)$ is proportional to the density of eigenstates of energy ω available to a particle of momentum k, while $\tilde{\Psi}_k(\omega)$ is proportional to the occupation of such states by particles of this momentum. Thus Eq. (10.32) has the usual form of a single-particle equilibrium distribution law if $f(\omega)$ is a function of ω/θ ($\theta=$ temperature) appropriate to the statistics of the particle.[30] In a later paper, we shall deduce distribution laws of this form directly from a condition of statistical equilibrium under small perturbations in the coupling among systems in a collection, without appealing to probability distributions in the space of the eigenstates (such as the grand canonical distribution).

11. TURBULENCE DYNAMICS

The problem of turbulence dynamics serves to illustrate the application of our methods to equations of motion which are nonlinear in the dynamic variables. In order to keep the formalism as simple as possible, we shall work here with the one-dimensional scalar analog to the Navier-Stokes equation proposed by Burgers.[31] The treatment of the Navier-Stokes equation for an incompressible fluid, which we shall discuss briefly, does not differ in essentials.

Burgers' equation is

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}\right) u(x,t) = -u(x,t)\frac{\partial u(x,t)}{\partial x} = -\frac{1}{2}\frac{\partial}{\partial x}[u(x,t)]^2.$$
$$(11.1)$$

The function $u(x,t)$ may be interpreted as the velocity of an infinitely compressible fluid, of constant kinematic viscosity ν, executing one-dimensional motion. If $\nu=0$, the quantities

$$\int_{-\infty}^{\infty} u(x,t)dx, \quad \frac{1}{2}\int_{-\infty}^{\infty} [u(x,t)]^2 dx$$

are both constants of motion. We shall call them "momentum" and "energy," respectively. [This is not their accurate meaning, however, on the basis of the interpretation just suggested for $u(x,t)$.]

If an infinitesimal forcing term $\delta f(x,t)$ is added to the right side of Eq. (11.1) for $t>t_0$, the response is

$$\delta u(x,t) = \int_{t_0}^t ds \int_{-\infty}^{\infty} dy G_{[1]}(x,t|y,s)\delta f(y,s),$$

where the infinitesimal Green's function $G_{[1]}(x,t|x',t')$

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[29] V_k necessarily is real and nonnegative.

[30] We may note that $\int d^3k G_k(-i\theta^{-1}) = G(x, -i\theta^{-1}|x,0)$ is the mean one-particle partition function per unit volume.

[31] J. M. Burgers, Advances in Appl. Mech. 1, 171 (1948).

obeys

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)G_{[\ ]}(x,t\,|\,x',t')$$

$$=-\frac{\partial}{\partial x}[u(x,t)G_{[\ ]}(x,t\,|\,x',t')], \quad (11.2)$$

$$G_{[\ ]}(x,t'\,|\,x',t')=\delta(x-x').$$

In correspondence to the procedure followed in Secs. 4, 8, and 10, let us take a collection of systems with velocity fields $u_{[n]}(x,t)$ and Green's functions

$$G_{[n,m]}(x,t\,|\,x',t'),$$

pass to the collective representation, and consider, instead of Eqs. (11.1) and (11.2), model equations of the form

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)u_\alpha(x,t)$$

$$=-\tfrac{1}{2}M^{-\frac{1}{2}}\sum_\beta\phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x}[u_\beta(x,t)u_{\alpha-\beta}(x,t)], \quad (11.3)$$

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)G_{\alpha,\gamma}(x,t\,|\,x',t')$$

$$=-M^{-\frac{1}{2}}\sum_\beta\phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x}[u_\beta(x,t)G_{\alpha-\beta,\gamma}(x,t\,|\,x',t')],$$

$$G_{\alpha,\gamma}(x,t'\,|\,x',t')=\delta_{\alpha,\gamma}\delta(x-x'). \quad (11.4)$$

As before, the ϕ's are independent of x and t and the same for all ensemble-realizations of the collection.

We shall impose upon the ϕ's the three conditions

$$\phi_{\alpha,\beta,\alpha-\beta}=\phi_{\alpha,\alpha-\beta,\beta}, \quad \phi_{-\alpha,-\beta,-\alpha+\beta}=\phi_{\alpha,\beta,\alpha-\beta}{}^*, \quad (11.5)$$

$$\phi_{\alpha-\beta,-\beta,\alpha}=\phi_{\alpha,\beta,\alpha-\beta}{}^*.$$

The first is a symmetry convention. It does not restrict the dynamics.[32] The second insures that Eq. (11.3) preserves the property

$$u_\alpha(x,t)=u_{-\alpha}{}^*(x,t)$$

and, therefore, the reality of the $u_{[n]}(x,t)$. The third is identical with Eq. (4.4). It insures that

$$\tfrac{1}{2}\sum_n\int_{-\infty}^\infty[u_{[n]}(x,t)]^2dx=\tfrac{1}{2}\sum_\alpha\int_{-\infty}^\infty|u_\alpha(x,t)|^2dx$$

(a)   (b)

Fig. 16. Vertex representations for the turbulence problem.

[32] We have, in fact, assumed this condition in writing Eq. (11.4).

is a constant of motion, if $\nu=0$. The property

$$\frac{d}{dt}\int_{-\infty}^\infty u_{[n]}(x,t)dx=0$$

follows from Eq. (11.3) for any values of the ϕ's, provided the $u_{[n]}(x,t)$ vanish at $x=\infty$. In correspondence to Eq. (8.7) we shall also require

$$\phi_{\mu,\lambda,\sigma}=1 \quad (\mu,\ \lambda\text{ or }\sigma=0). \quad (11.6)$$

The additional conditions which the ϕ's now satisfy imply only minor modifications in the diagrammatic representation introduced in Sec. 4. Let us associate with $\phi_{\mu,\lambda,\sigma}$ and $\phi_{\mu,\lambda,\sigma}{}^*$ the vertices shown in Figs. 16(a) and 16(b), respectively. Then the rules for associating diagrams with $C_{2n;p}(\alpha,\beta,\alpha-\beta)$ and $C_{2n;p}$ are identical with those given in Sec. 4, if dashed lines are replaced by solid lines everywhere.[33] We shall assume hereafter that Eq. (4.13) is satisfied.

Let us take

$$u_{[n]}(x,t_0)=\bar u(x,t_0)+u_{[n]}{}'(x,t_0), \quad (11.7)$$

where the initial values $u_{[n]}{}'(x,t_0)$ are identically distributed, with zero mean, for each n and statistically independent for different n. In correspondence to Eq. (3.8), it then follows that the moments of the $u_\alpha(x,t_0)$ vanish unless the sum of indices is zero. Now suppose that Eq. (11.3) is solved by iteration. From this property of the initial value moments, and the combination rule for indices in Eq. (11.3), we find

$$\langle u_\alpha(x,t)u_\beta(x',t')\cdots\rangle=0 \quad (\alpha+\beta+\cdots\neq0). \quad (11.8)$$

Similarly, the iteration solution of Eq. (11.4) yields

$$\langle G_{\alpha,\gamma}(x,t\,|\,x',t')\rangle=0 \quad (\alpha\neq\gamma). \quad (11.9)$$

It follows immediately from Eq. (11.8) that

$$\langle u_{[n]}(x,t)\rangle=\langle M^{-\frac{1}{2}}u_0(x,t)\rangle$$

for all n. In correspondence to the similar result cited in Sec. 8, it can be shown that $M^{-\frac{1}{2}}u_0(x,t)$ is a sharp quantity $(M\to\infty)$. Let us write

$$u_{[n]}(x,t)=\bar u(x,t)+u_{[n]}{}'(x,t),$$
$$\bar u(x,t)=\langle u_{[n]}(x,t)\rangle. \quad (11.10)$$

We shall call $\bar u(x,t)$ and $u_{[n]}{}'(x,t)$ the mean and fluctuating fields, respectively. By identifying $M^{-\frac{1}{2}}u_0(x,t)$ with $\bar u(x,t)$ in the limit $M\to\infty$, and noting Eq. (11.6), we may now rewrite Eqs. (11.3) and (11.4) in the form

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)\bar u(x,t)+\bar u(x,t)\frac{\partial\bar u(x,t)}{\partial x}$$

$$=-\tfrac{1}{2}M^{-1}\sum_\alpha{}'\frac{\partial}{\partial x}\langle u_\alpha(x,t)u_{-\alpha}(x,t)\rangle, \quad (11.11)$$

[33] The additional symmetry properties expressed by Eq. (11.5) result in an ambiguity in the formal expressions for the $C_{2n;p}$ given by the rules in Sec. 4. There is, however, no ambiguity in value.

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)u_\alpha(x,t)+\frac{\partial}{\partial x}[\bar{u}(x,t)u_\alpha(x,t)]$$

$$=-\tfrac{1}{2}M^{-1}\sum_\beta{}''\phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x}[u_\beta(x,t)u_{\alpha-\beta}(x,t)]$$

$$(\alpha\neq 0), \quad (11.12)$$

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)G_{\alpha,\gamma}(x,t|x',t')+\frac{\partial}{\partial x}[\bar{u}(x,t)G_{\alpha,\gamma}(x,t|x',t')]$$

$$=-M^{-\frac12}\sum_\beta{}'\phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x}[u_\beta(x,t)G_{\alpha-\beta,\gamma}(x,t|x',t')],$$

$$G_{\alpha,\gamma}(x,t'|x',t')=\delta_{\alpha,\gamma}\delta(x-x'), \quad (11.13)$$

where $\sum_\beta{}'$ implies that $\beta=0$ is to be omitted and $\sum_\beta{}''$ implies that both $\beta=0$ and $\alpha-\beta=0$ are to be omitted. It should be noted that Eq. (11.13) has the same form for $\alpha\neq 0$ and $\alpha=0$. Equations (11.11) and (11.12) are coupled equations which determine the evolution of the mean and fluctuating fields.

Now let us assume that the distribution of the initial values $u_{[n]}'(x,t_0)$ is multivariate Gaussian. It can then be shown from the iteration solutions of Eqs. (11.12) and (11.13), using arguments similar to those in Secs. 4 and 8, that $\langle u_\alpha(x,t)u_{-\alpha}(x',t')\rangle$ is independent of α $(\alpha\neq 0)$ and that $\langle G_{\alpha,\alpha}(x,t|x',t')\rangle$ is independent of α (all α). Then it follows from Eqs. (11.8) and (11.9) that

$$\langle u_{[n]}'(x,t)u_{[m]}'(x',t')\rangle=\delta_{n,m}U(x,t;x',t'),$$
$$U(x,t;x',t')=\langle u_\alpha(x,t)u_{-\alpha}(x',t')\rangle \quad (\alpha\neq 0),$$
$$\langle G_{[n,m]}(x,t|x',t')\rangle=\delta_{n,m}G(x,t|x',t'), \quad (11.14)$$
$$G(x,t|x',t')=\langle G_{\alpha,\alpha}(x,t|x',t')\rangle.$$

In correspondence to our previous results, $G_{\alpha,\alpha}(x,t|x',t')$ is statistically sharp $(M\to\infty)$, and the covariances satisfy

$$\langle u_\alpha(x,t)u_{-\alpha}(x',t')u_\beta(y,s)u_{-\beta}(y',s')\cdots\rangle$$
$$=U(x,t;x',t')U(y,s;y',s')\cdots+O(M^{-1}) \quad (11.15)$$
$$(\alpha,\beta,\cdots\neq 0, |\alpha|\neq|\beta|\neq\cdots)$$

[cf. Eq. (8.14)]. It follows from Eq. (11.14) that $U(x,t;x',t')$ has the symmetry property

$$U(x,t;x',t')=U(x',t';x,t). \quad (11.16)$$

From Eqs. (11.12) and (11.13) we find that $U(x,t;x',t')$ and $G(x,t|x',t')$ satisfy equations of the form

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)U(x,t;x',t')+\frac{\partial}{\partial x}[\bar{u}(x,t)U(x,t;x',t')]$$

$$=S(x,t;x',t'), \quad (11.17)$$

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)G(x,t|x',t')+\frac{\partial}{\partial x}[\bar{u}(x,t)G(x,t|x',t')]$$

$$=H(x,t|x',t'), \quad (11.18)$$

$$G(x,t'|x',t')=\delta(x-x'),$$

where it follows from the iteration solutions, with Gaussian $u_{[n]}'(x,t_0)$, that $S(x,t;x',t')$ and $H(x,t|x',t')$ have the forms

$$S(x,t;x',t')=\sum_n\sum_p^{\text{irr}}C_{2n;p}\xi_{2n;p}(x,t;x',t'), \quad (11.19)$$

and

$$H(x,t|x',t')=\sum_n\sum_p^{\text{irr}}C_{2n;p}\zeta_{2n;p}(x,t|x',t').^{21} \quad (11.20)$$

To complete the set of equations, we may rewrite Eq. (11.11) in the form

$$\left(\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x^2}\right)\bar{u}(x,t)+\bar{u}(x,t)\frac{\partial\bar{u}(x,t)}{\partial x}$$

$$=-\frac{1}{2}\frac{\partial}{\partial x}U(x,t;x,t). \quad (11.21)$$

Equation (11.21) is the balance equation for mean "momentum" density and Eq. (11.17) for $x',t'=x,t$ is the balance equation for the mean "energy" density in the fluctuating field.

The functions $\xi_{2n;p}(x,t;x',t')$ and $\zeta_{2n;p}(x,t|x',t')$ may be determined by the variational procedure of Sec. 4, using Eq. (11.15) and the statistical sharpness of $G_{\alpha,\alpha}(x,t|x',t')$. The results for $\xi_{2;1}(x,t;x',t')$ and $\zeta_{2;1}(x,t|x',t')$ are

$$\xi_{2;1}(x,t;x',t')$$

$$=\frac{1}{2}\frac{\partial}{\partial x}\int_{t_0}^{t'}ds\int_{-\infty}^{\infty}dyG(x',t'|y,s)\frac{\partial}{\partial y}[U(x,t;y,s)]^2$$

$$+\frac{\partial}{\partial x}\int_{t_0}^{t}ds\int_{-\infty}^{\infty}dyG(x,t|y,s)$$

$$\times\frac{\partial}{\partial y}[U(x,t;y,s)U(x',t';y,s)], \quad (11.22)$$

$$\zeta_{2;1}(x,t|x',t')=\frac{\partial}{\partial x}\int_{t_0}^{t}ds\int_{-\infty}^{\infty}dyG(x,t|y,s)$$

$$\times\frac{\partial}{\partial y}[U(x,t;y,s)G(y,s|x',t')]. \quad (11.23)$$

In general $\xi_{2n;p}(x,t;x',t')$ consists of a sum of terms each of which involves a $(2n-1)$-fold space-time integration over a product of $2n-1$ factors G and $n+1$

factors U. The terms comprising $\zeta_{2n;p}(x,t|x',t')$ each involve a $(2n-1)$-fold integration over a product of $2n$ factors G and n factors U.

We shall illustrate the variational procedure in the present case by outlining the analysis for $\xi_{2;1}(x,t; x',t')$. In correspondence to Eq. (8.16), we may write, for the present problem,

$$S(x,t; x',t') = \sum_\beta{}''\langle S_{\alpha,\beta,\alpha-\beta}(x,t; x',t')\rangle \quad (\alpha \neq 0),$$

$$S_{\alpha,\beta,\alpha-\beta}(x,t; x',t') = -\tfrac{1}{2}M^{-\frac{1}{2}}\phi_{\alpha,\beta,\alpha-\beta}\frac{\partial}{\partial x}[u_\beta(x,t)$$

$$\times u_{\alpha-\beta}(x,t)u_{-\alpha}(x',t')].$$

Then the iteration solution yields

$$\langle S_{\alpha,\beta,\alpha-\beta}(x,t; x',t')\rangle$$

$$= M^{-1}\sum_{n=1}^{\infty}\sum_{p}^{\text{irr}} C_{2n;p}(\alpha, \beta, \alpha-\beta)\xi_{2n;p}(x,t; x',t') \quad (11.25)$$

$$(\alpha, \beta, \alpha-\beta \neq 0, \quad |\alpha| \neq |\beta| \neq |\alpha-\beta|),$$

which corresponds to Eq. (8.21), and is the basis for Eq. (11.19). Now consider the variation Eq. (4.17). By using Eq. (11.5) several times, we find

$$\Delta\phi_{\alpha-\beta,-\beta,\alpha} = \Delta\phi_{\alpha-\beta,\alpha,-\beta} = \Delta\phi_{\beta,-\alpha+\beta,\alpha}$$

$$= \Delta\phi_{\beta,\alpha,-\alpha+\beta} = \Delta\phi_{-\alpha,-\beta,-\alpha+\beta} = \Delta\phi_{-\alpha,-\alpha+\beta,-\beta}.$$

Hence, recalling Eq. (11.3), we find

$$\Delta u_\beta(x,t) = \int_{t_0}^{t} ds \int_{-\infty}^{\infty} dy\, G_{\beta,\beta}(x,t|y,s)$$

$$\times\left\{-M^{-\frac{1}{2}}\Delta\phi_{\beta,\alpha,-\alpha+\beta}\frac{\partial}{\partial y}[u_\alpha(y,s)u_{-\alpha+\beta}(y,s)]\right\}, \quad (11.26)$$

to order $M^{-\frac{1}{2}}$, with expressions of the same type for $\Delta u_{\alpha-\beta}(x,t)$ and $\Delta u_{-\alpha}(x',t')$.[34] These results correspond to Eq. (8.24). It is important to note that the perturbation terms are $O(M^{-\frac{1}{2}})$, so that the infinitesimal Green's functions correctly may be used to find the induced variations. Now we may express

$$\Delta\langle S_{\alpha,\beta,\alpha-\beta}(x,t; x',t')\rangle$$

to $O(M^{-1})$ in correspondence to Eq. (8.25), reduce the averages by using Eq. (11.15) and the sharpness of the $G_{\alpha,\alpha}$, and appeal to the analog of Eq. (8.23). Thereby, we obtain the result Eq. (11.22).

The random coupling model for the present problem is obtained by assigning the ϕ's as in Sec. 5, but with the additional constraints Eqs. (11.5) and (11.6). It is

clear that these constraints do not affect Eq. (5.3) in the limit $M \to \infty$. Hence, we have

$$S(x,t; x',t') = \xi_{2;1}(x,t; x',t'),$$
$$H(x,t|x',t') = \zeta_{2;1}(x,t|x',t') \quad (11.27)$$

for this model. These relations, together with Eqs. (11.16)–(11.18) and Eqs. (11.21)–(11.23), form a closed set which determine $\bar{u}(x,t)$, $U(x,t; x',t')$, and $G(x,t|x',t')$ in terms of the initial functions $\bar{u}(x,t_0)$ and $U(x,t_0; x',t_0)$.

The most essential difference between the present equations and the analogous ones for the random potential problem given in Sec. 10 is that $G(x,t|x',t')$ is not independent of $U(x,t; x',t')$ and $\bar{u}(x,t)$ in the present case; all three quantities now must be determined simultaneously. A further consequence of the nonlinearity is that $\bar{u}(x,t)$ does not have an expression analogous to Eq. (10.18). The Green's function $G(x,t|x',t')$ can only describe the propagation of infinitesimal disturbances $\delta\bar{u}(x',t')$. In general, $\bar{u}(x,t) \neq 0$ even if $\bar{u}(x,t_0) = 0$ everywhere.

The Navier-Stokes equation for the velocity $\mathbf{u}(\mathbf{x},t)$ of an infinite incompressible fluid of kinematic viscosity ν may be written, after elimination of the pressure term,[3] in the tensor form

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u_i(\mathbf{x},t) = -\tfrac{1}{2}P_{imn}(\nabla)[u_m(\mathbf{x},t)u_n(\mathbf{x},t)],$$
$$(11.28)$$

where

$$P_{imn}(\nabla) = \left(\delta_{i,m} - \nabla^{-2}\frac{\partial^2}{\partial x_i \partial x_m}\right)\frac{\partial}{\partial x_n}$$

$$+ \left(\delta_{i,n} - \nabla^{-2}\frac{\partial^2}{\partial x_i \partial x_n}\right)\frac{\partial}{\partial x_m},$$

and

$$\nabla^{-2}f(\mathbf{x}) \equiv -(4\pi)^{-1}\int |\mathbf{x}-\mathbf{y}|^{-1}f(\mathbf{y})d^3y$$

for any f. We may treat the incompressible turbulence problem in direct analogy to the foregoing analysis by taking a collection of flow systems with individual velocity fields $u_i^{[n]}(\mathbf{x},t)$ and considering the model equation[35]

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u_i^{\alpha}(\mathbf{x},t)$$

$$= -\tfrac{1}{2}M^{-\frac{1}{2}}P_{inm}(\nabla)\sum_\beta \phi_{\alpha,\beta,\alpha-\beta}[u_n^{\beta}(\mathbf{x},t)u_m^{\alpha-\beta}(\mathbf{x},t)],$$

$$(11.29)$$

where the $u_i^{\alpha}(x,t)$ are the collective velocity fields.

The final equations for the random coupling model

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[34] Only terms involving solely diagonal elements of the Green's function matrix are of leading order $(M \to \infty)$. Thus, for example, the variation in $u_{-\alpha}$ induced by the perturbation terms in the equation of motion for u_β does not contribute in the limit.

[35] We use superscripts to label individual and collective quantities here, in order to avoid possible confusion with tensor indices.

which result from Eq. (11.29) are similar to those for Burgers' equation, but more complicated. In the case of homogeneous turbulence, they take their simplest form when transformed to correspond to a representation of the velocity field by spatial Fourier modes. They are then identical with equations for homogeneous turbulence derived previously by a different method.[36] The earlier derivation exploited the fact that the Fourier amplitudes of a homogeneous field have statistical properties which closely resemble those of the collective coordinates used in the present paper (cf. Sec. 3). Unlike the present approach, which involves no geometrical symmetry restrictions and which may be extended to fully bounded flows, the earlier treatment is valid only in the homogeneous case. A discussion of the energy dynamics of the random coupling model is given in Sec. 4 of the first reference cited in footnote 36.

We wish, finally, to give a very brief discussion of turbulent convection, which will serve to illustrate a point raised at the end of Sec. 9. Let $\psi(\mathbf{x},t)$ represent the zero-mean fluctuations in the concentration of marked particles carried along with an incompressible turbulent flow which obeys Eq. (11.28). Then $\psi(\mathbf{x},t)$ satisfies

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\psi(\mathbf{x},t)=-u_i(\mathbf{x},t)\frac{\partial\psi(\mathbf{x},t)}{\partial x_i}, \quad (11.30)$$

where κ is the molecular diffusivity. If $\psi_{[n]}(\mathbf{x},t)$ and $\psi_\alpha(\mathbf{x},t)$ represent, respectively, the individual and collective fields for a collection of flows, the model equation corresponding to Eq. (11.30) is

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\psi_\alpha(\mathbf{x},t)=-M^{-\frac{1}{2}}\sum_\beta\phi_{\alpha,\beta,\alpha-\beta}$$
$$\times u_i^\beta(\mathbf{x},t)\frac{\partial\psi_{\alpha-\beta}(\mathbf{x},t)}{\partial x_i}, \quad (11.31)$$

where Eqs. (11.5) and (11.6) are satisfied.

The random coupling model equations which result from Eq. (11.31), under Gaussian initial conditions of the form we have taken before, are

$$\left(\frac{\partial}{\partial t}-\kappa\nabla_x^2+\bar{u}_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\right)G(\mathbf{x},t\,|\,\mathbf{x}',t')$$

$$=\int_{t'}^t ds\int d^3y\,U_{ij}(\mathbf{x},t;\,\mathbf{y},s)\frac{\partial G(\mathbf{x},t\,|\,\mathbf{y},s)}{\partial x_i}\frac{\partial G(\mathbf{y},s\,|\,\mathbf{x}',t')}{\partial y_j},$$

$$G(\mathbf{x},t'\,|\,\mathbf{x}',t')=\delta(\mathbf{x}-\mathbf{x}'), \quad (11.32)$$

and

$$\left(\frac{\partial}{\partial t}-\kappa\nabla_x^2+\bar{u}_i(\mathbf{x},t)\frac{\partial}{\partial x_i}\right)\Psi(\mathbf{x},t;\,\mathbf{x}',t')$$

$$=\frac{\partial}{\partial x_i}\int_{t_0}^{t'}ds\int d^3y\,U_{ij}(\mathbf{x},t;\,\mathbf{y},s)G(\mathbf{x}',t'\,|\,\mathbf{y},s)$$

$$\times\frac{\partial\Psi(\mathbf{x},t;\,\mathbf{y},s)}{\partial y_j}+\frac{\partial}{\partial x_i}\int_{t_0}^t ds\int d^3y\,U_{ij}(\mathbf{x},t;\,\mathbf{y},s)$$

$$\times G(\mathbf{x},t\,|\,\mathbf{y},s)\frac{\partial\Psi(\mathbf{x}',t';\,\mathbf{y},s)}{\partial y_j}, \quad (11.33)$$

where

$$\bar{u}_i(\mathbf{x},t)=\langle u_i^{[n]}(\mathbf{x},t)\rangle,$$

$$U_{ij}(\mathbf{x},t;\,\mathbf{x}',t')=\langle u_i^{[n]\prime}(\mathbf{x},t)u_j^{[n]\prime}(\mathbf{x}',t')\rangle,$$

$$\Psi(\mathbf{x},t;\,\mathbf{x}',t')=\langle\psi_{[n]}(\mathbf{x},t)\psi_{[n]}(\mathbf{x}',t')\rangle,$$

and $G(\mathbf{x},t\,|\,\mathbf{x}',t')$ is the mean diagonal Green's function for the concentration field.[37] We have assumed $\langle\psi_{[n]}(\mathbf{x},t)\rangle=0$, a condition which is preserved by the equations of motion. A detailed study of the consequences of these equations when the velocity field is statistically homogeneous has been made by Roberts,[38] who derives the equations for this case by methods related to those of the references cited in footnote 36. Another case has been discussed by the present author.[39]

In accord with the discussion in Sec. 9, the random coupling equations for turbulent convection involve only $\bar{u}_i(\mathbf{x},t)$ and the covariance tensor $U_{ij}(\mathbf{x},t;\,\mathbf{x}',t')$, regardless of the distribution of the fluctuating part of the velocity field. Suppose, now, we ask how the higher statistical structure of the velocity field can be incorporated in higher stochastic models for the convection problem. If this structure were known explicitly, we could, in principle, insert the associated cumulants in the non-Gaussian terms, of the type in Eq. (9.4), which contribute in the higher models. An alternative procedure is to assume Gaussian initial conditions for both the concentration field and the fluctuating velocity field and then treat Eqs. (11.29) and (11.31) as a simultaneous set, making the ϕ's identical in the two equations. The sequence of higher models for this problem would commence with that of Sec. 7, and the non-Gaussian diagrams would never arise. The assumption of Gaussian initial conditions often may be physically plausible, particularly if the flow has persisted long

[36] R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959); see also, *Second Symposium on Naval Hydrodynamics*, edited by R. Cooper (United States Government Printing Office, Washington, 1960). The equations corresponding to the random coupling model are called the "direct-interaction approximation" equations in these papers.

[37] $G(\mathbf{x},t\,|\,\mathbf{x}',t')d^3x$ is the probability that a marked particle introduced at $\mathbf{x}'$, t' is in d^3x at $\mathbf{x},t$.

[38] P. H. Roberts (to be published). [Issued also as Rept. HSN-2, Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University (1960).]

[39] R. H. Kraichnan, in *Hydrodynamic Instability*, Vol. 13, *Proceedings of Symposia in Applied Mathematics*, edited by G. Birkhoff (American Mathematical Society, Providence, Rhode Island, to be published).

enough that the higher statistical structure of the velocity field is determined principally by the dynamics rather than by the cumulants of the initial distribution.

APPENDIX A

The second relation in Eq. (3.12) may be written

$$\langle G_{\alpha,\alpha}(t)G_{\alpha,\alpha}{}^*(t)\rangle - \langle G_{\alpha,\alpha}(t)\rangle\langle G_{\alpha,\alpha}{}^*(t)\rangle = O(M^{-1}), \quad \text{(A.1)}$$

where we use $\langle G_{\alpha,\alpha}(t)\rangle = G(t)$. Let the left-hand side of Eq. (A.1) be expanded by iteration of Eq. (4.6). In each term of the expansion of $G_{\alpha,\alpha}(t)$ or $G_{\alpha,\alpha}{}^*(t)$, the sum of indices of the b factors is zero [cf. argument leading to Eq. (4.7)]. Consequently, $\langle G_{\alpha,\alpha}(t)\rangle\langle G_{\alpha,\alpha}{}^*(t)\rangle$ consists of terms of the form

$$(-i)^{r-s}M^{-(r+s)/2}\sum_{\beta,\cdots,\mu,\gamma,\cdots,\sigma}(\text{product of }\phi\text{'s})$$
$$\times\langle b_\beta\cdots b_\mu b_{-\beta-\cdots-\mu}\rangle\langle b_\gamma\cdots b_\sigma b_{-\gamma-\cdots-\sigma}\rangle t^{r+s}/r!s!, \quad \text{(A.2)}$$

where there are r factors b in the first average and s in the second. For each such term there will be a corresponding term

$$(-i)^{r-s}M^{-(r+s)/2}\sum_{\beta,\cdots,\mu,\gamma,\cdots,\sigma}(\text{product of }\phi\text{'s})$$
$$\times\langle b_\beta\cdots b_\mu b_{-\beta-\cdots-\mu}b_\gamma\cdots b_\sigma b_{-\gamma-\cdots-\sigma}\rangle t^{r+s}/r!s! \quad \text{(A.3)}$$

in the expansion of $\langle G_{\alpha,\alpha}(t)G_{\alpha,\alpha}{}^*(t)\rangle$, where the product of ϕ's is identical for given indices β, $\cdots$, γ, $\cdots$. Let the ϕ's be bounded. Then the difference of Eqs. (A.2) and (A.3) is bounded in magnitude by

$$M^{-(r+s)/2}|(\text{product of }\phi\text{'s})|_{\max}\sum_{\beta,\cdots,\mu,\gamma,\cdots,\sigma}$$
$$\times|\langle b_\beta\cdots b_\mu b_{-\beta-\cdots-\mu}\rangle\langle b_\gamma\cdots b_\sigma b_{-\gamma-\cdots-\sigma}\rangle$$
$$-\langle b_\beta\cdots b_\mu b_{-\beta-\cdots-\mu}b_\gamma\cdots b_\sigma b_{-\gamma-\cdots-\sigma}\rangle| t^{r+s}/r!s!. \quad \text{(A.4)}$$

It now follows straightforwardly from Eq. (9.3) (we take the general non-Gaussian case) that Eq. (A.4) is $O(M^{-1})$ if $|(\text{product of }\phi\text{'s})|_{\max}$ is independent of M ($M\to\infty$). Similar analysis establishes Eqs. (8.14) and (11.15), if the iteration solutions of Eqs. (8.9) and (11.12), respectively, are used to express the equations in terms of the parameters and initial values, whose statistical properties are prescribed. In these solutions it is convenient to let the mean amplitudes remain in the expansions as parameters.

The significance of Eq. (3.12) was discussed in the text. Equations (8.14) and (11.15) also may be understood qualitatively as consequences of the fact that the dynamical behavior of a collective degree of freedom is determined ($M\to\infty$) by interaction with an infinite number of other degrees of freedom: The dynamical coupling with any given few of the other degrees of freedom is infinitesimal in the limit, and this implies a corresponding weakness of statistical dependence.